Part IV

Week 3A: Representation Theory of Groups

Representation theory in a nutshell

- Objects can be understood by their actions on simpler objects
- Representation of finite groups studies actions of the group on finite dimensional vector spaces
- Representations of groups “linearizes” the group structure
- Representations are occasionally required for deeper positive characteristic results

Definitions

- A **representation** of a finite group G is a group homomorphism from G to the automorphism group of a finite dimensional vector space V
- The vector space with basis G has an obvious representation where g takes the hth basis vector to hg. This is called the **regular representation.**
- In fact, it has a natural ring structure called the **group ring**, kG
- A **module** is a ring homomorphism from kG to the endomorphism ring of a finite dimensional vector space
- Modules and representations are the same thing
- Even simpler, we can just specify one invertible matrix per generator of the group

Example

- We give the example of S_3 acting on a three dimensional space as a permutation of the standard basis. Define $\pi : S_3 \to GL(3,k)$ by:

$$[1,2,3] \mapsto \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix} \quad [2,1,3] \mapsto \begin{pmatrix} . & 1 & . \\ 1 & . & . \\ . & . & 1 \end{pmatrix} \quad [1,3,2] \mapsto \begin{pmatrix} 1 & . & . \\ . & . & 1 \\ . & 1 & . \end{pmatrix}$$

$$[3,1,2] \mapsto \begin{pmatrix} . & . & 1 \\ 1 & . & . \\ . & 1 & . \end{pmatrix} \quad [2,3,1] \mapsto \begin{pmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{pmatrix} \quad [3,2,1] \mapsto \begin{pmatrix} . & . & 1 \\ . & 1 & . \\ 1 & . & . \end{pmatrix}$$

- The module structure takes

$$\sum_{g \in S_3} \alpha_g e_g \in k^G \mapsto \sum_{g \in S_3} \alpha_g \pi(g) \in End(k^3)$$

Submodules

- A **submodule** of $\pi : kG \to End(V)$ is a restriction $\sigma : kG \to End(W)$ where $\sigma(w) = \pi(w) \in W \leq V$ for all $w \in W$.

- Everyone abbreviates the name of π to "V" and the name of σ to "W" to make it easier to talk about elements

- If we form a basis of V by extending a basis of W then then the matrices have the form

$$\pi(g) = \begin{pmatrix} \sigma(g) & \cdot \\ B(g) & C(g) \end{pmatrix}$$

- $B(g)$ is "trash" but $C(g)$ defines a representation $C : G \to Aut(V/W)$ called a **quotient module**

Direct sums

- If $V = U \oplus W$ with both U, W submodules, then write a basis for V by concatenating bases for U and W.
- In this basis we have the even simpler form

$$\pi(g) = \begin{pmatrix} \sigma(g) & \cdot \\ \cdot & C(g) \end{pmatrix}$$

  where σ defines the submodule structure of U and C defines the quotient module structure of $V/U \cong W$

- This is nicer because $B(g) = 0$ can be ignored. $B(g)$ describes the interaction of U and W and makes life difficult if it is nonzero
- We say that V is **decomposable** or a **direct sum** in this case where $B(g) = 0$
- If $U \ncong W$ then virtually every question about V is simply the disjoint union of the same question for U and W

Atomic power

- Many areas of science try to break things down into pieces which cannot be broken down further
- A module is **irreducible = simple** if it has no nonzero proper submodules
- A module is **indecomposable** if it is not a direct sum of two nonzero proper submodules
- To understand the module theory means to understand the indecomposable modules
- This is possible for human minds iff the Sylow p-subgroup is cyclic, dihedral, or p does not divide $|G|$
- In the latter case, indecomposable = irreducible and the theory is trivial (direct product of fields)
- This is our case and why character theory usually suffices

Example again

- Our representation of S_3 on k^3 is not irreducible, the subspace spanned by $\langle 1, 1, 1\rangle$ forms a submodule
- We extend this to a basis

$$e_1 = \langle 1, 1, 1\rangle \quad e_2 = \langle 0, 1, -1\rangle \quad e_3 = \langle -1, 0, -1\rangle$$

- Let $g = [2, 1, 3]$, then $e_1\pi(g) = e_1$, $e_2\pi(g) = \langle 1, 0, -1\rangle = -2e_1 - 2e_2 - 3e_3$, and $e_3\pi(g) = \langle 0, -1, -1\rangle = -2e_1 + e_2 - 2e_3$, so in the new basis

$$\pi(g) \mapsto \begin{pmatrix} \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} . & . \end{bmatrix} \\ \begin{bmatrix} -2 \\ -2 \end{bmatrix} & \begin{bmatrix} -2 & -3 \\ 1 & -2 \end{bmatrix} \end{pmatrix}$$

- Submodule is top left, quotient module is bottom right, but trashy bottom left is the cohomology

Example yet again

- We know from theory the cohomology vanishes except for $p = 3$, but we want a basis that shows this.

- Let $f_1 = \langle 1, 1, 1\rangle, f_2 = \langle 1, -1, 0\rangle, f_3 = \langle 0, 1, -1\rangle$ which is a basis if the characteristic is not 3, and we get the new matrices:

$$[1,2,3] \mapsto \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix} \quad [2,1,3] \mapsto \begin{pmatrix} 1 & . & . \\ . & \text{-1} & . \\ . & 1 & 1 \end{pmatrix} \quad [1,3,2] \mapsto \begin{pmatrix} 1 & . & . \\ . & 1 & 1 \\ . & . & \text{-1} \end{pmatrix}$$

$$[3,1,2] \mapsto \begin{pmatrix} 1 & . & . \\ . & \text{-1} & \text{-1} \\ . & 1 & . \end{pmatrix} \quad [2,3,1] \mapsto \begin{pmatrix} 1 & . & . \\ . & . & 1 \\ . & \text{-1} & \text{-1} \end{pmatrix} \quad [3,2,1] \mapsto \begin{pmatrix} 1 & . & . \\ . & . & \text{-1} \\ . & \text{-1} & . \end{pmatrix}$$

- Notice the block structure $\sigma(g) = [1]$ and $C(g)$ the bottom right 2×2 corner.

Module theory for the symmetric group

- Roughly speaking we do exactly the same thing as we did with characters:

  1. Find the permutation representations on the cosets of Young subgroups (row stabilizers of Young tableaux = centralizers of Young tabloids)
  2. Use Gram-Schmidt to break down the representations into direct sums of previously known simple representations and one copy of the unique new simple representation

- Gram-Schmidt requires subtraction!
- Subtracting modules is hard.
- Polytabloids (=linear combinations of row-equivalence classes of Young tableaux) form a natural basis of that unique new simple module, called the **Specht module**

Conclusion

- Representations = modules linearize a group
- Irreducible = simple ones form building **block**s
- Indecomposables form the real buildings (or pyramids at least)
- For us, indecomposable = irreducible
- Breaking down representations into irreducibles is very hard, so we need to use the polytabloids to form a basis

THE END