PT-Groups and defect two subnormal subgroups

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Quasi-normality

- For a subgroup $H \leq G$, with G a finite group:
- **Normal** iff $gH = Hg$ for all **elements** g in G.
  Written $H \trianglelefteq G$
- **Permutable** iff $KH = HK$ for all **subgroups** K of G.
  Written $H \operatorname{per} G$
- **Sylow-permutable** iff $PH = HP$ for all **Sylow subgroups** P of G.
  Written $H \operatorname{s-per} G$
- **Subnormal** of defect at most n iff there are subgroups $K_1, \ldots, K_n$ with $H \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_n = G$.
  Written $H \trianglelefteq\trianglelefteq_n G$
- normal $\Rightarrow$ permutable $\Rightarrow$ Sylow-permutable $\Rightarrow$ subnormal

Transitive relation groups

- Subnormality is always transitive, $H \trianglelefteq\trianglelefteq K \trianglelefteq\trianglelefteq G \Rightarrow H \trianglelefteq\trianglelefteq G$
- If normality is transitive, the group is called a **T-group**
- If permutability is transitive, the group is called a **PT-group**
- If Sylow permutability is transitive, the group is called a **PST-group**
- T-group $\subseteq$ PT-group $\subseteq$ PST-group $\subseteq$ All groups

Subnormality of defect two

- Normality is transitive exactly when $H \trianglelefteq\trianglelefteq G \Rightarrow H \trianglelefteq G$
- However, it clearly suffices to check only defect two:
- Normality is transitive exactly when $H \trianglelefteq \mathbf{K} \trianglelefteq G \Rightarrow H \trianglelefteq G$

- Sylow-permutability is transitive exactly when $H \trianglelefteq\trianglelefteq G \Rightarrow H\,\text{s-per}\,G$
- Ballester-Bolinches, Esteban-Romero, and Ragland (2007) consider just-non-PST groups to show that it suffices to check defect two:
- Sylow-permutability is transitive exactly when $H \trianglelefteq \mathbf{K} \trianglelefteq G \Rightarrow H\,\text{s-per}\,G$

Should it work in general?

- Obvious to conjecture that a similar argument works for PT
- However, the classification of just-non-PT differs somewhat than for just-non-PST
- Also, two-step-subnormals being “somewhat” permutable, does not imply that all subnormals are as permutable
- Indeed there is an example of a group in which all two-step-subnormal subgroups permute with all Sylow 2-subgroups, but not all subnormal subgroups do:

A counterexample

- Let P be the extra-special group of order 125 and exponent 5,

$$P = \langle x, y, z | x^5 = y^5 = z^5 = 1, [x, y] = z, [x, z] = [y, z] = 1 \rangle$$

- P has commuting automorphisms a, b of order $2, 3$ with "matrices"

$$a = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- The two-step subnormal subgroups of the semidirect product $G = \langle a, b, x, y, z \rangle$ are either maximal in P or normal in G
- Maximals of P are normalized by P and a, so permute with all conjugates of $\langle a \rangle$, so are Sylow-2-permutable
- But $\langle x \rangle$ is three-step subnormal, and does not permute with $\langle a \rangle$
- Permutability of three-step subnormals is not simply automatic

Just-non-PT PST groups

- Let $\mathcal{PT}_2$ be the groups such that two-step subnormal subgroups are permutable
- $q\mathcal{PT}_2 = \mathcal{PT}_2 \subseteq \mathcal{PST}$, so a group of smallest order in $\mathcal{PT}_2 \setminus \mathcal{PT}$ is a PST group that is just-non-PT
- Let G be PST and all its proper quotients are PT,
let H be PST and just-non-PT
- Robinson has shown G is PT iff $G/G^{(\infty)}$ is PT,
so $H^{(\infty)} = 1$ and H is soluble.
- Agrawal has shown soluble G is PT iff $G/\gamma_\infty(G)$ is PT,
so $\gamma_\infty(H) = 1$ and H is nilpotent.
- Nilpotent G is clearly PT iff its normal Sylow subgroups are PT,
so H is a p-group
- A p-group is PT if and only if it is "modular", and
just-non-modular p-groups were classified by Longobardi

Longobardi groups are not $\mathcal{PT}_2$

- Longobardi showed that a just-non-modular p-group H is either

  1. a central product of a non-modular p-group M of order p^3 with some group from an explicit list, or
  2. is from a specific family of groups:
     For $0 < s < n \leq j + s$, and $s \geq 2$ for $p = 2$, let

$$K(p,n,s,j) = \langle w, a, b | w^p = a^{p^n} = 1, b^{p^j} = a^{p^{n-s}}, \\ a^b = a^{1+p^s}, a^w = a^{1+p^{n-1}}, b^w = b \rangle$$

- In the central product case, M has a non-permutable subgroup X of order p, which is two step subnormal in M. Since the rest of the group is a central product, X is two-step-subnormal in H, but still not permutable in H, so no such group is in $\mathcal{PT}_2$

- The "K" case seemed harder:

The "K" case

- In the "K" case, GAP experiments showed it not to be in $\mathcal{PT}_2$ for "small" parameters, and in fact a pattern emerged:
- The subgroup $W = \langle w \rangle$ is two-step-subnormal, but not permutable, so no "K" is in $\mathcal{PT}_2$
- For p odd, the subgroup $L = \langle g \rangle = \langle b^{p^{j+s-n}} a^{-1} \rangle$ does not permute with W
- Easy to see that $WL = LW$ iff $[w, g] \in L$, by the semidirect product
- Easy to see that if $[w, g] = g^t$, then $0 \equiv t \mod p^{n-s}$, by checking mod $\langle a \rangle$
- Almost easy to see that $g^{p^s} = 1$:
- W has a cyclic derived subgroup, so is a regular p-group, so $(BA)^t = 1$ iff $B^t A^t = 1$
- Set $BA = g$ with $B = b^{p^{j-(n-s)}}$, $A = a^{-1}$, $t = p^{n-s}$

An application

- Theorem: A group is T, PT, PST iff every subnormal subgroup of defect two is normal, permutable, Sylow permutable

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- If a subgroup H permutes with its conjugates, $HH^g = H^g H$ for all g in G, then it is said to be conjugate-permutable.
- Szep showed that any conjugate-permutable subgroup is subnormal
- Every two-step-subnormal subgroup is conjugate permutable: $H \trianglelefteq K \trianglelefteq G$ gives $H^g \trianglelefteq K$ and normal subgroups permute
- Corollary: A group is T, PT, PST iff every conjugate-permutable subgroup is normal, permutable, Sylow permutable.

Current work

- Finish an argument that handles "permutable sensitive" classification simultaneously avoiding the "K" case for $p = 2$
- Handle the appropriate local versions in a uniform manner
- Express the classification of just-non-PST succinctly, and similarly to PST + just-non-PT

THE END