PST, just-non-PT groups

Jack Schmidt

University of Kentucky

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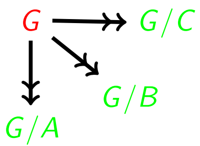
Outline

- Intro to just-non-? groups
- Examples of just-non-PT groups
- Classifications of just-non-PT groups
- Application and a surprising example
- Current work on PST, just-non-PT

Just-non-? = Groups whose proper quotients are ?

Let $\mathcal{X}$ be a class of groups we want to understand



Study groups that are **just** barely **not** in $\mathcal{X}$

just-non-$\mathcal{X}$ = not in $\mathcal{X}$, but every proper quotient is

Just-non-trivial groups = simple groups

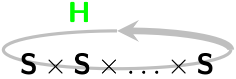
1960: Newman: countable, soluble, just-non-**abelian**
1970: McCarthy and Wilson: just-non-**finite**
1973: Robinson: soluble, just-non-**T**
1982: Longobardi: finite, nilpotent, just-non-**PT**
2009: Working on finite, PST, just-non-**PT**

Examples: finite, just-non-cyclic

Examples of finite, non-cyclic, but every proper quotient is cyclic:

- Klein four, or any **elementary abelian group of rank** 2
- any dihedral group of order $2p$
- A_4 the alternating group on four points, or any $\mathbf{AGL(1, q)}$
- S_5, or any symmetric group on at least five points
- the **automorphism group of** $M_{11} \times M_{11}$

Classification: finite, just-non-cyclic



- Three types:

  1. Elementary abelian p-group of rank 2
  2. $H(n,p) \leq AGL(1,p^k)$ for $n > 1$, p prime, $k = Order(p \mod n)$
  3. $S^k \leq G \leq Aut(S^k) = Aut(S) \wr Sym(k)$ with G/S^k cyclic and acting transitively on $\{1,\ldots,k\}$

- Second type takes an element of order n in $GF(p^k)^\times$ acting on the additive group of $GF(p^k)$.

- Standard example of just-non-$\mathcal{X}$: Semidirect of $\mathcal{X}$-group acting faithfully and irreducibly on some other group

Collapse of quasi-normality

Several classes of groups are defined based on various ideas of "normality" being equal in a group:

- **T-group** = subnormal subgroups are normal
  $H \trianglelefteq \ldots \trianglelefteq G \implies H \trianglelefteq G$, that is $gH = Hg$ for all $g \in G$

- **PT-group** = subnormal subgroups are permutable
  $H \trianglelefteq \ldots \trianglelefteq G \implies \langle g \rangle H = H \langle g \rangle$ for all $g \in G$

- **PST-group** = subnormal subgroups are Sylow permutable
  $H \trianglelefteq \ldots \trianglelefteq G \implies PH = HP$ for all Sylows $P \leq G$

Examples of PT-groups

- $T \implies PT \implies PST$

- Every abelian group is T, PT, and PST
- Every nilpotent group is PST
- Nilpotent T groups are abelian or Hamiltonian ($Q_8 \times O \times E$)

- Dihedral groups of order $2p$ are T, but not abelian
- $p \ltimes p^2$ is PT, but not T
- Dihedral group of order 8 is PST, but not PT

Example of just-non-PT groups

- Extraspecial groups like D_8 are finite, nilpotent, just-non-abelian, just-non-T, just-non-PT, PST
- $D_8 \curlyvee C_{2^n}$ is also finite, nilpotent, just-non-abelian, just-non-T, just-non-PT, PST
- $S_3 \times C_3$ is finite, supersoluble, just-non-T, just-non-PT, just-non-PST
- $(C_3 \ltimes C_9) \ltimes 7^3$ is finite, soluble, not supersoluble, just-non-PT, just-non-PST

Classifications

- Finite, soluble, PST, just-non-PT groups $=$ just-non-modular p-groups of Longobardi 1982 (reduction shown to me by Matt Ragland)
- Finite, supersoluble, non-nilpotent, just-non-PT groups $\stackrel{?}{=}$ some just-non-T types of Robinson 1973 (preliminary)
- Finite, soluble, non-supersoluble, just-non-PT groups $=$ "standard type": PT-group $\ltimes$ faithful simple module of dim ≥ 2
- Finite, insoluble, PST, just-non-PT seem within reach

Application

- T-group: $H \trianglelefteq N \trianglelefteq G \implies H \trianglelefteq G$
- PST-group: $H \trianglelefteq N \trianglelefteq G \implies H$ S-per G
- Soluble, PT-group: $H \trianglelefteq N \trianglelefteq G \implies H$ per G

- Soluble, PT-group proof used soluble, PST, just-non-PT groups
- Ramon Esteban-Romero found a nice counterexample for **insoluble** PT-groups

An interesting example

- Just-non-PT, p-group P with $M \trianglelefteq P$ such that $H \leq M, H \text{ per } P \iff H \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_k = P$

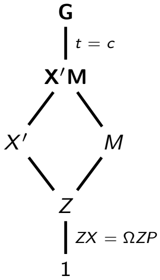
  $P = \langle a, b, c : a^{p^{k+1}} = b^{p^k} = c^p = [c, a] = 1, [a, b] = a^p, [c, b] = a^{p^k} \rangle$,

  $M = \langle a, b \rangle$

- $|ZX| = |X/X'| = p$ and X'/ZX simple

  $X'/ZX = PSL(p, q)$ for $1 \equiv q \mod p$, $X = \langle t, X' \rangle$

- The central product of X and P contains a subgroup $G = \langle X, M, tc^{-1} \rangle$
- G is finite, insoluble, PST, just-non-PT, and every subnormal subgroup of defect at most k is permutable



Local classification

Define:

$$N_p^o = \{G : H \leq O_p(G) \implies [O^p(G), H] \leq H\}$$

$$P_p^o = \{G : H \leq O_p(G) \implies H \text{ per } G\}$$

$$N_p^* = \{G : G/M \in N_p^o,\ \forall M \trianglelefteq G\}$$

$$P_p^* = \{G : G/M \in P_p^o,\ \forall M \trianglelefteq G\}$$

Then:

PST $= N_p^*$ for all p and simple chief factors

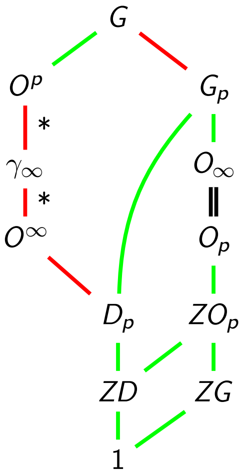
PT $=$ PST and P_p^* for all p

PST, just-non-PT are just-non-P_p^o for exactly one prime p and P_q^* for all other primes q

Lattice and partial results

Suppose: G PST, just-non-PT group, O_p non-abelian, G_p a sylow of G, $D = O^\infty(G)$

- G/D is soluble PT, so has standard form
- $G/DO_\infty \leq Out(D)$, so has standard form
- $O_p(G) = O_\infty(G)$, "solvable part" is easy
- D is central product of quasi-simples, G_p acts diagonally on it
- G itself should be a nice central product of just-non-PT with D quasi-simple and PT
- $O^p(G) \stackrel{?}{=} O^\infty(G)$, at least in the "central irreducible" case?



Summary

- Just-non-X groups are interesting when X is interesting
- PT used to have the same classifications as T and PST, but now is special
- We should understand just-non-PT groups, especially insoluble
- We need both concrete and conceptual classifications

THE END