Sylow structure of finite groups

Jack Schmidt

University of Kentucky

September 2, 2009

ABSTRACT: Probably the most powerful results in the theory of finite groups are the Sylow theorems. Those who have studied for the Algebra prelim know they are used to prove groups of certain orders cannot be simple. In fact, the Sylow subgroups control the structure of a finite group much more strongly than just deciding non-simplicity. This talk will describe work from the 19th and 20th centuries on the extent to which Sylow subgroups determine a group up to isomorphism.

Outline

- Hölder era results on square-free groups
- Burnside era results on cyclic Sylow subgroups
- Burnside and later results on Sylow 2-subgroups
- Fancier classifications of Brauer and Gorenstein-Walter

Groups by their order

- **Lagrange's theorem**: the order of an element or a subgroup of a group divides the order of the group
- **Cauchy's theorem**: if a prime divides the order of a group, then the group has an element of that order
- There are 18 groups of order up to 10, 1048 of order up to 100, 11 758 814 of order up to 1000, 49 910 529 484 of order up to 2000
- Most groups have order 2^n; the more times a prime divides, the more groups there are
- What if no prime squared divides the order?
  Then we can describe them completely.
- Next: Examples of nice groups

The nonabelian group of order 6

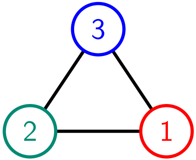
- The symmetric group on three points
  $\langle (1,2),(1,2,3) \rangle$

- A semidirect product of 2 acting on 3
  $\langle a, b : a^2 = b^3 = 1, ba = abb \rangle \cong 2 \ltimes 3$

- The Coxeter group of type A_2
  $\langle a, c : a^2 = c^2 = 1, cac = aca \rangle$

- A matrix group over the integers, or over any nonzero ring
  $\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) \rangle$

- The group of symmetries of an equilateral triangle
  $\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \rangle$



The nonabelian group of order 6

The important representations are

- As a permutation group:
  $\langle(1,2),(1,2,3)\rangle$.
- A semidirect product of 2 acting on 3:
  $\langle a, b : a^2 = b^3 = 1, ba = abb\rangle$.
- A matrix group over the field of three elements:
  $\langle\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$.

Dihedral groups of order 2p

- As a permutation group:
  $\langle (1, p-1)(2, p-2)\cdots(\frac{p-1}{2}, \frac{p+1}{2}), (1, 2, \ldots, p)\rangle$

- As a semidirect product of 2 with p:
  $\langle a, b : a^2 = b^p = 1, ba = ab^{p-1}\rangle$

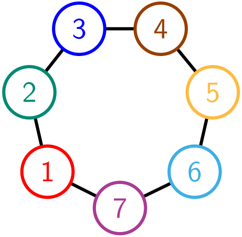
- As a Coxeter group of type $I_2(p)$:
  $\langle a, c : a^2 = c^2 = 1, c(ac)^{(p-1)/2} = a(ca)^{(p-1)/2}\rangle$

- As a matrix group over the finite field of size p:
  $\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\rangle$

- As a symmetry group of a regular p-gon:
  $\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos(2\pi/p) & \sin(2\pi/p) \\ -\sin(2\pi/p) & \cos(2\pi/p) \end{pmatrix}\rangle$

Dihedral groups of order 14

- As a permutation group:
  $\langle (1,6)(2,5)(3,4), (1,2,3,4,5,6,7) \rangle$
  $\langle (1,6)(2,5)(3,4), (1,2,3,4,5,6,7) \rangle$



- As a semidirect product of 2 with 7:
  $\langle a, b : a^2 = b^7 = 1, ba = ab^6 \rangle$
- As a Coxeter group of type $I_2(7)$:
  $\langle a, c : a^2 = c^2 = 1, cacacac = acacaca \rangle$
- As a matrix group over the finite field of size 7:
  $\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$
- As a symmetry group of a regular heptagon:
  $\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos(2\pi/7) & \sin(2\pi/7) \\ -\sin(2\pi/7) & \cos(2\pi/7) \end{pmatrix} \rangle$

Important representations of D2p

- As a permutation group: $\langle (1, p-1)(2, p-2)\cdots(\frac{p-1}{2}, \frac{p+1}{2}), (1, 2, \ldots, p)\rangle$
- As a semidirect product of 2 with p: $\langle a, b : a^2 = b^p = 1, ba = ab^{p-1}\rangle$
- As a matrix group over the finite field of size p: $\langle \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$

Important representations of D14

- As a permutation group: $\langle (1, 6)(2, 5)(3, 4), (1, 2, 3, 4, 5, 6, 7)\rangle$
- As a semidirect product of 2 with 7: $\langle a, b : a^2 = b^7 = 1, ba = ab^6\rangle$
- As a matrix group over the finite field of size 7: $\langle \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$

Set z to be a primitive root mod p

AGL(1,p)

- As a permutation group, it is $N_{S_p}(\langle(1,2,...,p)\rangle)$, $\langle(1,z,z^2,z^3,\ldots,z^{p-2}) \mod p,(1,2,\ldots,p)\rangle$
- As a semidirect product of $p-1$ with p: $\langle a,b : a^{p-1}=b^p=1, ba=ab^z\rangle$
- As a matrix group over the finite field of size p: $\langle \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$

AGL(1,7)

- As a permutation group: $\langle(1,3,2,6,4,5),(1,2,3,4,5,6,7)\rangle$
- As a semidirect product of 6 with 7: $\langle a,b : a^6=b^7=1, ba=ab^3\rangle$
- As a matrix group over the field of 7 elements: $\langle \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$

Here m, n, z are positive integers, $1 = \gcd((z-1)n, m)$, and $1 \equiv z^n \mod n$

Metacyclic groups M(m,n,z)

- As a permutation group: $\langle (1, z, z^2, z^3, \ldots, z^{n-1}) \cdots (a, az, az^2, \ldots, az^{n-1}) \mod m, (1, 2, \ldots, m) \rangle$
- As a semidirect product of n with m: $\langle a, b : a^n = b^m = 1, ba = ab^z \rangle$
- As a matrix group over the ring $\mathbb{Z}/m\mathbb{Z}$: $\langle \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$

Examples

- $M(3,7,2) = \langle (1,2,4)(3,6,5), (1,2,3,4,5,6,7) \rangle$
- $M(2,7,6) = D_{2 \cdot 7} = \langle (1,6)(2,5)(3,4), (1,2,3,4,5,6,7) \rangle$
- $M(6,7,3) = AGL(1,7) = \langle (1,3,2,6,4,5), (1,2,3,4,5,6,7) \rangle$

Classification of square-free groups

If G is a group whose order is not divisible by the square of any prime:

- **Frobenius 1893**: G is solvable
- **Hölder 1895**: G is isomorphic to a metacyclic group $M(m, n, z)$
- Proof: Let $H = [G, G]$, $K = [H, H]$, and $L = [K, K]$. Both H/K and K/L are abelian of square-free order, so cyclic. Since K/L is cyclic, its automorphism group is cyclic, and so $H = [G, G]$ centralizes K/L. But then $(H/L)/(K/L)$ is cyclic and K/L is contained in the center of H/L, so we can apply a well-known prelim problem: H/L is abelian. That means $K = L$, and since G is solvable, $K = L = 1$ and $H = [G, G]$ itself is cyclic.

  Take b to be a generator of $[G, G]$, and a to be the preimage of a generator of $G/[G, G]$, then $a^{-1}ba = b^z$ for some z.
- Burnside 1905 generalized this, but first we need Sylow subgroups:

Sylow subgroups

- A p-**group** is a group whose order is a power of p
- Theorem: If H is a p-subgroup of G and p divides $[G : H]$, then p divides $[N_G(H) : H]$
- Every p-subgroup of G is contained in a **Sylow** p-subgroup P, that is P is a p-group and $[G : P]$ is coprime to p
- All Sylow p-subgroups are conjugate
- Burnside 1905 showed that if all Sylow subgroups are cyclic, then the group is isomorphic to the metacyclic group $M(m, n, z)$

How much can a single Sylow tell us?

- What if we only know the Sylow 2-subgroup P?
- Well, for any group N of odd order $P \times N$ has Sylow 2-subgroup P
- For any $f : P \to \mathrm{Aut}(N)$, also $P \ltimes_f N$ has Sylow 2-subgroup P
- While this is a lot of groups, for some P they are the **only** groups
- If $G = P \ltimes N$ for a Sylow p-subgroup P,
  then we say G is **p-nilpotent** and we consider it well known
- If G has order twice an odd number, that is $|G| = 4k + 2$,
  then G has a subgroup H of index $2\ldots$, so G is 2-nilpotent.

Cyclic Sylow 2-subgroup $\Rightarrow$ 2-nilpotent

- Burnside showed that if a group has a cyclic Sylow 2-subgroup, then it is 2-nilpotent
- **Proof:** Suppose G has a cyclic Sylow 2-subgroup of order 2^n. If $n = 0$, there is nothing to prove. Consider the permutation action of G on itself. An element of order 2^n is a product of an odd number of 2^n-cycles, so is an odd permutation. Hence $H = G \cap A_n \neq G$ is a subgroup of index 2 in G. H has a cyclic Sylow 2-subgroup of order 2^{n-1}, so by induction $H = P_0 \ltimes N$ for P_0 a cyclic subgroup of order 2^{n-1} and N a normal subgroup of H of odd order. Since N is the unique maximal subgroup of odd order in H, N is also normal in G. Taking P to be any Sylow 2-subgroup containing P_0, one has $G = P \ltimes N$.
- If a group has a cyclic Sylow 2-subgroup, then it is solvable.
- All finite simple groups have a cyclic Sylow p-subgroup for some p

Other Sylows that force 2-nilpotency

- Say that P is **2-nilpotent forcing** if the only groups containing P as a Sylow 2-subgroup are 2-nilpotent
- Burnside: If P is **cyclic**, then P is 2-nilpotent forcing
- Burnside: If P is **unbalanced abelian** (decomposition into cyclic summands has no repeated sizes), then P is 2-nilpotent forcing. Example: $16 \times 4 \times 2$
- Yoshida 1978: If $\mathrm{Aut}(P)$ is a 2-group and P has no quotient isomorphic to D_8, then P is 2-nilpotent forcing
- Conversely, if P is 2-nilpotent forcing, then $\mathrm{Aut}(P)$ is a 2-group

Not all groups are 2-nilpotent!

The alternating group on four points (A4)

- As permutation group: $\langle (1,2,3),(1,2)(3,4),(1,4)(2,3)\rangle$
- As semidirect product $3 \ltimes (2 \times 2)$: $\langle a, x, y : a^3 = x^2 = y^2 = 1, xa = ay, ya = axy, yx = xy\rangle$
- As matrix group, over the field of 4 elements, with z a primitive 3rd root of unity: $\langle \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)\rangle$

The normal subgroups of A4 are 1, K4, and A4. The only one of odd order is 1. Hence **A4 is not 2-nilpotent**.

Fancier version of A4

The **Eisenstein integers** form a nice ring, $\mathbb{Z}[\omega]$ where $\omega^2+\omega+1=0$ and ω is a primitive 3rd root of unity. The ideal $P=2R$ is prime and R/P is a field with 4 elements. R/P^n is a ring with 4^n elements. Additively it is $\mathbb{Z}/2^n\mathbb{Z}\times\mathbb{Z}/2^n\mathbb{Z}$ with basis $\{1,\omega\}$.

Define a new group $A_4(n)$

- As a semidirect product $3 \ltimes (2^n \times 2^n)$: $\langle a, x, y : a^3 = x^{2^n} = y^{2^n} = 1, xa = ay, ya = axy, yx = xy\rangle$
- As a matrix group over R/P^n: $\langle \left(\begin{smallmatrix}\omega & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & \omega\\ 0 & 1\end{smallmatrix}\right)\rangle \leq AGL(1, R/P^n)$

The normal subgroups of $A_4(n)$ are $K_4(i)=\langle x^{2^i}, y^{2^i}\rangle$ for $1\leq i\leq n$ and $A_4(n)$. The only one of odd order is $K_4(n)=1$, so $A_4(n)$ **is not 2-nilpotent**.

A classification for $P = 2^n \times 2^n$

- Let N be the largest normal subgroup of G of odd order.
- Note $A_4(n)$ has Sylow 2-subgroup $2^n \times 2^n$
- **Brauer 1964**: If $n \geq 2$ and G has Sylow 2-subgroup $P = 2^n \times 2^n$, then $G/N \cong P$ or $G/N \cong A_4(n)$
- G is 2-nilpotent **iff** $G/N = P$
- For some P, we know G is 2-nilpotent, so we know $G/N = P$. For other P like $2^n \times 2^n$, this cannot work.
- For a given P, we want a list of all G/N!

More non 2-nilpotent groups

- The **special linear group** SL(n,q) is the group of all $n \times n$ matrices of determinant 1 over a field of size q
- $SL(2,3) = 3 \ltimes Q_8$ has the 2-group on the wrong side
- The **projective special linear group** PSL(n,q) is the quotient group of SL(n,q) by its center
- $PSL(2,3) = 3 \ltimes (2 \times 2) = A_4$ is not 2-nilpotent either
- $PSL(2,5) = A_5$ also has $P = 2 \times 2$ and is not 2-nilpotent
- These are more or less the only G/N with $P = 2 \times 2$:
- **Gorenstein&Walter 1965**: for $P = 2 \times 2$, either $G/N = P$ or

$$G/N = f_0 \ltimes d_0 \ltimes PSL(2, p^f)$$

for p, f_0, d_0 odd, f_0 divides f, d_0 divides $p^f - 1$, $p^f \equiv \pm 3 \mod 8$

Further classifications

- **Gorenstein&Walter 1965**: for **dihedral** P, either $G/N = P$ or $G/N = f_0 \ltimes d_0 \ltimes PSL(2, p^f)$ for p, f_0, d_0 odd, f_0 divides f and d_0 divides $p^f - 1$
- **Gorenstein's 1970 exercise**: for **quaternion** P, either $G/N = P$ or $G/N = f_0 \ltimes d_0 \ltimes SL(2, p^f)$ for p, f_0, d_0 odd, f_0 divides f and d_0 divides $p^f - 1$
- Many other classifications for nice P, but they restrict G (maybe G is generated by its Sylow 2-subgroups, or G is simple, etc.)
- These are combined inductively with classifications based on **involution centralizers**
- A substantial part of the classification of finite simple groups