Examples 4.1: Simplex algorithm

Vocabulary: There are several words that are used in chapters 3 and 4 that might not be familiar:

- **Tableau** - French word for table; plural is tableaux.
- **Linear programming problem** - The type of problem we have been solving, find the optimal, feasible solution given the variables, the constraints, and the objective function.
- **Optimal** - A flexible word that can mean "minimum" or "maximum" depending on the context (cost or profit)
- **Feasible** - We make our decisions subject to certain constraints. Well those of us who don't want to go to jail or end up bankrupt do. If a decision satisfies all the constraints (like not spending more money than we have, not having employees work more than their allotted shifts, not planting our crops on our neighbor's land, etc.), then the decision is feasible. If we violate any of the constraints, then the solution is infeasible.
- **Slack variables** - Clever names for how far we are from the border of the constraints. If we must have $x + 2y \leq 20$, then we might let $u = 20 - x - 2y$ be the slack variable that tells us how far from 20 we have managed to stay. If $u = 10$ we have plenty of room. If $u = 0$, then we are on the edge between allowed and not allowed. If $u = -10$ we have crossed severely into the land of going to jail.
- **Basic** - The method of corners tells us that we have an optimal feasible solution at a corner; that is an intersection of two edges; that means two of the slack variables are 0. Any solution found by just setting the right number of the variables to 0 is called a basic solution. It is crazy but true: if there is a feasible, optimal solution then there is a feasible, basic, optimal solution.
- **Simplex algorithm** - A method of deciding which variables we end up setting to 0 using the same ideas as from RREF.

Step 1: Set up the table

The first step is to convert the inequalities from the long form with variables, and plusses, and equals, to the shorter tabular form.

Question: Write down the tableau for the linear programming problem: Maximize $P = 3x + 4y + 5z$ subject to $2x + y \leq 10$, $2y + z \leq 20$, $2z + x \leq 30$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

Scratch work: Set up the slack variables and profit functions in a "variables = constant" way:

$$\begin{aligned} 2x + y + u &= 10 \\ 2y + z + v &= 20 \\ 2z + x + w &= 30 \\ -3x - 4y - 5z + P &= 0 \end{aligned}$$

Answer: Now convert the equations to "matrix" form like for exam 1:

x	y	z	u	v	w	P	RHS
2	1	0	1	0	0	0	10
0	2	1	0	1	0	0	20
1	0	2	0	0	1	0	30
-3	-4	-5	0	0	0	1	0

Step 2: Go pivoting!

The columns for u, v, and w look like an identity matrix. The x, y, z columns are like free variables in a badly sorted RREF. The basic solution for this table sets all the "free" variables to 0, so our first basic, feasible solution is $(x = 0, y = 0, z = 0)$. The bigger we make x, the bigger the profit, so this is not an optimal, feasible solution. How much bigger can we make x? In the top row, we currently have 10 slack, and every time we increase x by 1, we lose 2 slack. Hence we can increase x up to 5 as far as the first row is concerned. The second row does not care about x at all. The third row we have 30 slack and each x only uses 1 slack, so we could increase x up to 30 as far as the third row is concerned. We have to keep all the rows happy, so we change x to 5 and u to 0. x pivots in, u pivots out.

Step 2a: Choose a pivot column Choose the leftmost column with a negative entry in the bottom row as the pivot column. Any column with a negative entry is OK, but in this class we do the leftmost.

In the example, we choose the x column because it has a -3 in the bottom row and there are no other rows to its left.

Step 2b: Choose a pivot row In the pivot column there are numbers. If they are positive, then divide them into the right hand side. This is how far the pivot variable can be increased. Choose the row with the smallest increase (the most restrictive) to be the pivot row. In case of ties, choose the row that makes the leftmost variable pivot out.

In the example we find the ratios $10/2 = 5$ and (we skip the second row because of the 0 and) $30/1 = 30$ The smallest one is 5, so we choose the first row as the pivot row.

Step 2c: Do the row operations We want to do one column's worth of RREF. We make our pivot 1 by dividing by 2, and make everything else in the column 0 by using row operations:

$$\begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 20 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 30 \\ \hline -3 & -4 & -5 & 0 & 0 & 0 & 1 & 0 \end{array} \xrightarrow{\frac{1}{2}R_1} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 5 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 20 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 30 \\ \hline -3 & -4 & -5 & 0 & 0 & 0 & 1 & 0 \end{array} \xrightarrow[R_4+3R_1]{R_3-R_1} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 5 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 20 \\ 0 & -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & 1 & 0 & 25 \\ \hline 0 & -2.5 & -5 & \frac{3}{2} & 0 & 0 & 1 & 15 \end{array}$$

One step is done! Our new solution has pivots in the x, v, and w columns, and the y, z, and u columns are free. We set the free variables to 0 to get $(x = 5, y = 0, z = 0)$ as our new basic, feasible solution, and with slacks $(u = 0, v = 20, w = 25)$ and profit $P = 15$, up from $P = 0$. We read the value of x from the first row, the value of v from the second, the value of w from the third, and the value of P from the fourth.

Step 2d: Go back to step 2a unless you cannot find a pivot row, in which case go to step 3!

More example: Since y is free and since the larger the y the larger the profit, it is foolish to leave $y = 0$. We want to make y our pivot (it is the leftmost column in which the bottom row is negative). We have to stay feasible, so we see how far we can move y: the first row allows $5/\frac{1}{2} = 10$, the second row allows $20/2 = 10$, and the third row is negative so we skip it. We get to choose which variable pivots out; the first row has pivot x and the second has pivot z, so we choose the first row as the pivot row.

$$\begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 5 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 20 \\ 0 & -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & 1 & 0 & 25 \\ \hline 0 & -\frac{5}{2} & -5 & \frac{3}{2} & 0 & 0 & 1 & 15 \end{array} \xrightarrow{2R_1} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 & 20 \\ 0 & -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & 1 & 0 & 25 \\ \hline 0 & -\frac{5}{2} & -5 & \frac{3}{2} & 0 & 0 & 1 & 15 \end{array} \xrightarrow[R_4+\frac{5}{2}R_1]{\substack{R_2-2R_1 \\ R_3+\frac{1}{2}R_1}} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ -4 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 30 \\ \hline 5 & 0 & -5 & 4 & 0 & 0 & 1 & 40 \end{array}$$

Now we have pivots in the y, v, and w columns, so x, z, and u are free. We set the free variables to 0 to get our basic solution $(x = 0, y = 10, z = 0)$ with slacks $(u = 0, v = 0, w = 30)$ and profit $P = 40$. Now the larger x is, the SMALLER the profit since $P = 40 - 5x + 5z - 4u$. The only variable that increases profit is z, so we make z our new pivot column. The first row does not care about z. The second row says we cannot move z at all. The third row says we can move z by 15. The most restrictive is the second row, so we make it a pivot row. Notice that we will not end up changing z; this is called a degenerate pivot and is not awful, but does slow us down.

$$\begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ -4 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 30 \\ \hline 5 & 0 & -5 & 4 & 0 & 0 & 1 & 40 \end{array} \xrightarrow{\frac{1}{1}R_2} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ -4 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 30 \\ \hline 5 & 0 & -5 & 4 & 0 & 0 & 1 & 40 \end{array} \xrightarrow[R_4+5R_2]{R_3-2R_2} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ -4 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 9 & 0 & 0 & 4 & -2 & 1 & 0 & 30 \\ \hline -15 & 0 & 0 & -6 & 5 & 0 & 1 & 40 \end{array}$$

The y, z, and w columns have pivots, and the x, u, v columns are free. We set the free variables to 0 to get our basic solution $(x = 0, y = 10, z = 0)$ with slacks $(u = 0, v = 0, w = 30)$ and profit $P = 40$. Note that the profit, z and v all kept the same value, but that z and v switched places as far as being free. Now our profit is $P = 40 + 15x + 6u - 5v$, so we could increase profit by increasing x or u. We'll pick the leftmost one, and make x the pivot column. The first row restricts us to $10/2 = 5$, the second row has a negative so makes no restriction, and the third row restricts us to $30/9 \approx 3.3$, so the third row is the most restrictive. We choose the third row as the pivot row.

$$\begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ -4 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 9 & 0 & 0 & 4 & -2 & 1 & 0 & 30 \\ \hline -15 & 0 & 0 & -6 & 5 & 0 & 1 & 40 \end{array} \xrightarrow{\frac{1}{9}R_3} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ -4 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{4}{9} & -\frac{2}{9} & \frac{1}{9} & 0 & \frac{10}{3} \\ \hline -15 & 0 & 0 & -6 & 5 & 0 & 1 & 40 \end{array} \xrightarrow[R_4+15R_3]{\substack{R_1-2R_3 \\ R_2+4R_3}} \begin{array}{ccccccc|c} x & y & z & u & v & w & P & \text{RHS} \\ 0 & 1 & 0 & \frac{1}{9} & \frac{4}{9} & 0 & 0 & \frac{10}{3} \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{1}{9} & \frac{4}{9} & 0 & \frac{40}{3} \\ 1 & 0 & 0 & \frac{4}{9} & -\frac{2}{9} & \frac{1}{9} & 0 & \frac{10}{3} \\ \hline 0 & 0 & 0 & \frac{2}{3} & \frac{5}{3} & \frac{5}{3} & 1 & 90 \end{array}$$

Step 3: Read the answer Once the bottom row has no negative numbers, you are done doing step 2s. Now we just read the answer: We have pivots in the x, y, and z, so u, v, and w are free. We get our basic solution by setting the free variables to 0. We get $(x = \frac{10}{3}, y = \frac{10}{3}, z = \frac{40}{3}$ and slacks $(u = 0, v = 0, w = 0)$ and profit $P = 90$.

MA162

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Activity 4.1a: Setting up the table

1. Write down the table associated to the linear programming problem: Maximize $P = 3x + 4y$ subject to $x + y \leq 4$, $2x + y \leq 5$, $x \geq 0$, $y \geq 0$.

x	y	u	v	P	RHS

2. Write down the linear programming problem associated to the table:

x	y	u	v	P	RHS
5	6	1	0	0	20
9	7	0	1	0	25
-4	-8	0	0	1	0

Maximize: ________________________ subject to ____.

3. Write down the table associated to the linear programming problem: Maximize $P = 3x + 4u$ subject to $x + u \leq 4$, $2x + u \leq 5$, $x \geq 0$, $u \geq 0$.

x	y	u	v	P	RHS

4. Write down the linear programming problem associated to the table:

x	y	u	v	P	RHS
5	1	6	0	0	20
9	0	7	1	0	25
-4	0	-8	0	1	0

Maximize: ________________________ subject to ____.

Activity 4.1b: Pivoting

Circle the pivot column, then find the ratios, and then circle the pivot row (Sec 4.1, ex#11, p. 238):

$$\begin{array}{ccccc|cl}
x & y & u & v & P & \text{RHS} & \\
1 & 1 & 1 & 0 & 0 & 4 & \text{Ratio:}\underline{\hspace{5cm}} \\
2 & 1 & 0 & 1 & 0 & 5 & \text{Ratio:}\underline{\hspace{5cm}} \\
\hline
-3 & -4 & 0 & 0 & 1 & 0 &
\end{array}$$

Carry out the row operations: First row operation is: ______________________________

$$\begin{array}{ccccc|c}
x & y & u & v & P & \text{RHS} \\
 & & & & & \\
\hline
 & & & & &
\end{array}$$

Second and third row operations are: ______________________________

$$\begin{array}{ccccc|c}
x & y & u & v & P & \text{RHS} \\
 & & & & & \\
\hline
 & & & & &
\end{array}$$

Activity 4.1c: Reading the answer

Read the basic solution from the table (Sec. 4.1, ex#1, p. 237):

x	y	u	v	P	RHS
0	1	$\frac{5}{7}$	$-\frac{1}{7}$	0	$\frac{20}{7}$
1	0	$-\frac{3}{7}$	$\frac{2}{7}$	0	$\frac{30}{7}$
0	0	$\frac{13}{7}$	$\frac{3}{7}$	1	$\frac{220}{7}$

Decision: ($x =$ ________, $y =$ ________)

Result: $P =$ ________________

Slack: ($u =$ ________, $v =$ ________)

Read the basic solution from the table (Sec. 4.1, like ex#3, p. 237):

x	y	u	v	P	RHS
0	2	1	-2	0	2
1	2	0	2	0	4
0	-2	0	6	1	48

Decision: ($x =$ ________, $y =$ ________)

Result: $P =$ ________________

Slack: ($u =$ ________, $v =$ ________)

Adjusting: If we don't set the free variables to zero, what is the profit function? $P =$ ________________________________

Which variable should we increase? ________________

Read the basic solution from the table (Sec 4.1, like ex#7, p. 238):

x	y	z	s	t	u	v	P	RHS
1	0	0	2	0	-1	-2	0	6
0	0	0	-1	1	5	4	0	7
0	0	1	4	0	3	2	0	8
0	1	0	3	0	2	4	0	9
0	0	0	72	0	8	7	1	4920

Decision: ($x =$ ________, $y =$ ________, $z =$ ________)

Result: $P =$ ________________

Slack: ($s =$ ________, $t =$ ________, $u =$ ________, $v =$ ________)

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Quiz on 4.1: Simplex algorithm

Setup the table: Maximize $P = 6x + 3y$ subject to $x + y \leq 10$, $2x + y \leq 15$, $x \geq 0$, $y \geq 0$.

x	y	u	v	P	RHS

Do one complete pivot step (step 2):

x	y	u	v	P	RHS

Read the answer:

Decision: ($x =$ _______, $y =$ _______)

Result: $P =$ ______________

Slack: ($u =$ _______, $v =$ _______)