

Examples 4.2: Dual problems

Vocabulary:

- **Primal:** The original problem, usually a minimize LPP in sec 4.2
- **Dual:** A related but opposite problem with “the same” answer, usually a standard maximize LPP in sec 4.2, solved like in sec 4.1
- **LPP:** Linear Programming Problem, one of these “find the optimal value of a linear function subject to linear constraints” problems

Dualizing a minimization problem Write out the problem in short table form, flip it over, and write it back out in words:

Primal problem: Minimize $C = 3u + 4v$ subject to $5u + 6v \geq 7$, $8u + 9v \geq 10$, $u, v \geq 0$.

Scratch work: As a table this is:

$$\begin{array}{cc|c} 5 & 6 & 7 \\ 8 & 9 & 10 \\ \hline 3 & 4 & \end{array} \xrightarrow{\text{Dualize}} \begin{array}{cc|c} 5 & 8 & 3 \\ 6 & 9 & 4 \\ \hline 7 & 10 & \end{array}$$

Dual problem: Maximize $P = 7x + 10y$ subject to $5x + 8y \leq 3$, $6x + 9y \leq 4$, $x, y \geq 0$

This is solved exactly as in section 4.1, except stop at the final tableau. Reading the answer is done somewhat differently.

Reading the final tableau

The dual problem can be read as normal from the dual tableau. If a column for a variable is a pivot column (a basic variable), then find the row with the pivot in it, and the RHS of that row is the value of that basic variable. If a column does not contain a pivot, then we choose to set that variable to 0. The optimal value of the objective function is the bottom right entry.

The primal problem can be read even more quickly. For each column, the value in the bottom row is the value that is assigned to that variable for the primal solution. This is almost always different than the value assigned for the dual solution. The optimal value of the objective function is the same, the bottom right entry of the table.

If the primal problem is to minimize $C = 3u + 4v + 5w$ subject to $6u + 7v + 8w \geq 9$, $10u + 11v + 12w \geq 13$, then the dual problem is to maximize $P = 9x + 13y$ subject to $6x + 10y \leq 3$, $7x + 11y \leq 4$, $8x + 12y \leq 5$. The dual tableaux are:

$$\begin{array}{cccccc|c|c} x & y & u & v & w & P & \text{RHS} & (\text{Ratio}) & x & y & u & v & w & P & \text{RHS} \\ 6 & 10 & 1 & 0 & 0 & 0 & 3 & (3/6) & 1 & 5/3 & 1/6 & 0 & 0 & 0 & 1/2 \\ 7 & 11 & 0 & 1 & 0 & 0 & 4 & (4/7) & 0 & -2/3 & -7/6 & 1 & 0 & 0 & 1/2 \\ 8 & 12 & 0 & 0 & 1 & 0 & 5 & (5/8) & 0 & -4/3 & -4/3 & 0 & 1 & 0 & 1 \\ \hline -9 & -13 & 0 & 0 & 0 & 1 & 0 & & 0 & 2 & 3/2 & 0 & 0 & 1 & 9/2 \end{array} \rightarrow$$

The dual decision is $(x = 1/2, y = 0)$ resulting in $P = 9/2$ and slacks $(u = 0, v = 1/2, w = 1)$.

The primal decision is $(u = 3/2, v = 0, w = 0)$ resulting in $C = 9/2$ and slacks $(x = 0, y = 2)$.

We check that these decisions are feasible. First the dual:

- $6x + 10y (= 3) \leq 3$ with slack $u = 0$,
- $7x + 11y (= 3.5) \leq 4$ with slack $v = 0.5$,
- $8x + 12y (= 4) \leq 5$ with slack $w = 1$.

Then the primal:

- $6u + 7v + 8w (= 18/2) \geq 9$ with slack $x = 0$,
- $10u + 11v + 12w (= 30/2) \geq 13$ with slack $y = 2$

Name: _____

Section: _____

Date: 2010-03-02

Quiz on 4.2: Dual problems

Setup the dual table: Minimize $C = 3u + 4v$ subject to $u + v \geq 5, 2u + v \geq 6, u \geq 0, v \geq 0$

Hint: Dual problem is maximize $P = 5x + 6y$ subject to $x + 2y \leq 3, x + y \leq 4, x \geq 0, y \geq 0$

x	y	u	v	P		RHS	(Ratio)

Do one complete pivot step (step 2): Be sure to circle the pivot column, find the ratios, and circle the pivot row in the table above.

	x	y	u	v	P		RHS
R							
R							

Read the primal answer from the dual table:

Decision: $(u = \underline{\hspace{1cm}}, v = \underline{\hspace{1cm}})$

Result: $C = \underline{\hspace{1cm}}$

Slack: $(x = \underline{\hspace{1cm}}, y = \underline{\hspace{1cm}})$

Dual decision is $(x = 3, y = 0)$ with result $P = 15$ and slacks $(u = 0, v = 1)$

Name: _____

Section: _____

Date: 2010-03-02

Activity 4.2a: Setting up the table

1. Write down the dual table associated to the linear programming problem: Minimize $C = 3u + 4v$ subject to $u + v \geq 4$, $2u + v \geq 5$, $u \geq 0$, $v \geq 0$.

x	y	u	v	P	RHS
<hr/>					

2. Write down the linear programming problem associated to the dual table (assume that x, y are the dual variables and u, v are the primal variables):

x	y	u	v	P	RHS
5	6	1	0	0	20
9	7	0	1	0	25
-4	-8	0	0	1	0

Minimize: _____ subject to _____.

3. Write down the table associated to the linear programming problem: Minimize $C = 3u + 4v + 5w$ subject to $u + v \geq 4$, $v + w \geq 5$, $u \geq 0$, $v \geq 0$, $w \geq 0$.

x	y	u	v	w	P	RHS
<hr/>						

4. Write down the linear programming problem associated to the dual table:

x	y	u	v	w	P	RHS
5	6	1	0	0	0	20
9	7	0	1	0	0	25
10	11	0	0	1	0	35
-4	0	-8	0	0	1	0

Minimize: _____ subject to _____.

1. Read the answer for the primal problem from this final tableau for the dual problem (assume u, v, w were the primal variables and x, y the dual):

x	y	u	v	w	P	RHS
0	1	4	5	0	0	10
1	0	6	7	0	0	11
0	0	8	9	1	0	12
0	0	13	14	0	1	15

Dual: The maximum is $P = 15$ occurring at $(x = 11, y = 10)$ with slack variables $(u = 0, v = 0, w = 12)$

2. Read the answer for the primal problem from this dual tableau:

x	y	z	u	v	P	RHS
0	2	1	3	4	0	5
1	6	0	7	8	0	9
0	10	0	11	0	1	12

Dual: The maximum is $P = 12$ occurring at $(x = 9, y = 0, z = 5)$ with slacks $(u = 0, v = 0)$

3. In this problem, I'm only asking you to read the answer from the table. I'll work out the first part:

Question: Find the minimum of $C = 3u + 4v$ subject to $u + v \geq 4$, $2u + v \geq 5$, $u + 3v \geq 6$, $u \geq 0$, $v \geq 0$.

Solution: The dual problem is: maximize $P = 4x + 5y + 6z$ subject to $x + 2y + z \leq 3$, $x + y + 3z \leq 4$, $x \geq 0$, $y \geq 0$.

$$\begin{array}{c|c} \begin{array}{cccccc} x & y & z & u & v & P \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ \hline -4 & -5 & -6 & 0 & 0 & 1 \end{array} & \begin{array}{c} \text{RHS} \\ 3 \\ 4 \\ \hline 0 \end{array} \end{array} \begin{array}{c} (Ratio) \\ (3) \\ (4) \end{array} \rightarrow \begin{array}{c|c} \begin{array}{cccccc} x & y & z & u & v & P \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ \hline 0 & 3 & -2 & 4 & 0 & 1 \end{array} & \begin{array}{c} \text{RHS} \\ 3 \\ 1 \\ \hline 12 \end{array} \end{array} \begin{array}{c} (Ratio) \\ (3) \\ (1/2) \end{array} \rightarrow \begin{array}{c|c} \begin{array}{cccccc} x & y & z & u & v & P \\ 1 & 5/2 & 0 & 3/2 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 & 1/2 & 0 \\ \hline 0 & 2 & 0 & 3 & 1 & 1 \end{array} & \begin{array}{c} \text{RHS} \\ 5/2 \\ 1/2 \\ \hline 13 \end{array} \end{array}$$

From the final tableau for the dual problem, read the primal solution:

Primal: The minimum is $C =$ _____ occurring at $(u =$ _____, $v =$ _____).

Dual: The maximum is $P = 13$ occurring at $(x = 5/2, y = 0, z = 1/2)$ with slacks $(u = 0, v = 0)$.

You should check using the method of corners that this is correct. At the very least check that your answer is feasible (satisfies the constraints) and cannot be easily improved (say, by adding 1 to one of the variables).