2010-03-04

$$\begin{aligned} 1. \text{ Multiply} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 + 0 & 2 + 0 & 3 + 0 & 4 + 0 & 5 + 0 & 6 + 0 & 7 + 0 & 8 + 0 & 9 + 0 \\ 0 + 9 & 0 + 8 & 0 + 7 & 0 + 6 & 0 + 5 & 0 + 4 & 0 + 3 & 0 + 2 & 0 + 1 \\ 1 - 9 & 2 - 8 & 3 - 7 & 4 - 6 & 5 - 5 & 6 - 4 & 7 - 3 & 8 - 2 & 9 - 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \end{pmatrix}. \\ 2. \text{ Add } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + 2 \cdot \begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 + 2(3) & 2 + 2(2) & 3 + 2(1) \\ 4 + 2(6) & 5 + 2(5) & 6 + 2(4) \end{pmatrix} = \begin{pmatrix} 7 & 6 & 5 \\ 16 & 15 & 14 \end{pmatrix}. \end{aligned}$$

$$3. \text{ Invert } A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 3R_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$
so that $A^{-1} = \begin{pmatrix} 1 & -2 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$

4. McGruggin's Manufacturing produces farm gates and cattle guards in a time when everyone wants farm gates and cattle guards. He has more customers than he can handle, and he is having trouble running his factory to its full potential. The two products both require time on the welding line and the finishing machine. McGruggin realizes a \$20 profit on farm gates and a \$200 profit on cattle guards. The farm gates require 5 minutes on the welding line and 10 minutes on the finishing machine. The cattle guards require 60 minutes on the welding line, but only 30 minutes on the finishing machine. If McGruggin runs both the welding line and the finishing machine 12 hours a day, how many farm gates and how many cattle guards should he order to be made each day in order to maximize his profit?

Variables: Let f be the number of farm gates produced per day, and c be the number of cattle guards produced per day.

Inequalities: $5f + 60c \le 720$ (welding constraint), $10f + 30c \le 720$ (machine constraint), $f \ge 0$, $c \ge 0$ (standard inequalities)

Objective: Maximize P = 20f + 200c

1.

Slack variables: Let w be the minutes of welding time not used each day, and let m be the minutes of finishing machine time not used each day.

5. From the initial tableau, one step of the simplex algorithm arrives at the following intermediate tableau:

f	c	w	m	p	RHS	(Ratio)	1 1	f	С	w	m	p	RHS	(Ratio)
5	60	1	0	0	720	(144)	$R_1 - \frac{1}{2}R_2; R_3 + 2R_2; \frac{1}{10}R_2$	0	45	1	-1/2	0	360	(8)
10	30	0	1	0	720	(72)		1	3	0	1/10	0	72	(24)
-20	-200	0	0	1	0			0	-140	0	2	1	1440	

The first step of the simplex algorithm decided to make all the farm gates it could, f = 72, but still left 6 hours free for welding, w = 360. Circle the pivot column, calculate the ratios, circle the pivot row, and then apply the three row operations to write down the final tableau:

6. From the final tableau read the solution:

f	c	w	m	p	RHS
0	1	1/45	-1/90	0	8
1	0	-1/15	2/15	0	48
0	0	28/9	4/9	1	2560

Decision is (f = 48, c = 8) resulting in P = 2560, and slacks (w = 0, m = 0). In other words, each day McGruggin should plan to make 48 farm gates and 8 cattle guards, resulting in \$2560 profit each day, and leaving exactly 0 spare minutes of welding time or finishing machine time. In other words, he is using all of his resources, and achieving maximum profit.

7. Background: McGruggin's success has inspired his neighbor McFlugal. McFlugal wants to rent time on McGruggin's welding line and finishing machine, but knows that McGruggin would never rent his assets at a loss. McFlugal lets w denote the price per minute he'll offer to McGruggin for time on the welding line, and m denote the the price per minute he'll offer to McGruggin for time on the finishing machine. If McFlugal too needs 12 hours a day on those machines, his cost would be C = 720w + 720m. Now McFlugal knows McGruggin can earn \$20 profit using 5 minutes of welding time and 10 minutes of machine time, so McFlugal is going to have to offer at least enough so that $5w + 10m \ge 20$. Similarly, McGruggin can earn \$200 profit using 60 minutes of welding time and 30 minutes of machine time, so McFlugal is constrained by $60w + 30m \ge 200$. How much should McFlugal offer per minute of welding time and machine time in order to minimize his cost, while still presenting McGruggin with a sane choice? Simple version: Minimize C = 720w + 720m subject to $5w + 10m \ge 20$, $60w + 30m \ge 200$, $w \ge 0$, $m \ge 0$.

Once you dualize the problem, you get exactly the same tableau as before (though now you think of it as a "dual" tableau), and as you work out the simplex algorithm, everything goes the same so you get the same final tableau. Now read the answer of the primal problem from this final "dual" tableau.

Decision: (w = 28/9, m = 4/9) resulting in C = 2560 and slacks (f = 0, c = 0).

Interpretation: If McFlugal's offer is accepted, McGruggin will make the same amount of money each day, but will be making f = 0 farm gates and c = 0 cattle guards himself. Instead all of his money will be coming from the rent. If McGruggin rejects the offer, instead McGruggin will be making f = 48 farm gates and c = 8 cattle guards, but earning rent on w = 0 minutes of welding time and m = 0 minutes of machine time.

8. Instead of using the simplex algorithm, use the graphical method:

Simple version: Minimize C = 720w + 720m subject to $5w + 10m \ge 20$, $60w + 30m \ge 200$, $w \ge 0$, $m \ge 0$. Sketch and shade the region defined by the inequalities and use the method of corners to find the minimum value. Be sure to indicate why the minimum occurs at a corner.

