MA162: Finite mathematics

Jack Schmidt

University of Kentucky

March 2, 2011

Schedule:

- HW B3 is due Sunday, Mar 6th, 2011.
- Exam 2 is Monday, Mar 7th, 5:00pm-7:00pm.

Today we will cover breakdown in 4.1 and duality in 4.2

Exam 2: Overview

- 35% Ch 2, Matrix arithmetic (HWB1):
  - Sizes, Addition, subtraction, multiplication by a number, matrix-matrix multiplication, matrix to a power
  - Matrix inverse (2×2 and 3×3); Solving system given inverse
- 30% Ch 3, Two-variable linear programs (HWB2):
  - Setting up a linear programming problem
  - Solving using the graphical method
- 35% Ch 4, Simplex algorithm (HWB3):
  - Solving primal problems using simplex algorithm
  - **Solving dual problems using simplex algorithm**

4.2: Linear programming problems

- An LPP has three parts:
  - The variables (the business decision to be made)
  - The inequalities (the laws, constraints, rules, and regulations)
  - The objective (maximize profit, minimize cost)
- We used matrices to solve equations, paying attention to positives
- What if we have too many negatives?

4.2: Today's LPP as word problem

- MacFrugal next door wants to buy out MacD's business this season
- MacD could get \$150 for 1 acre, \$50, and 20hrs by planting crop A
- MacD could get \$200 for 1 acre, \$60, and 25hrs by planting crop B
- MacFrugal wants his 100 acres, \$6000, and 2400 labor hours
- How much should MacFrugal offer for each acre, each dollar of capital, and each hour of labor in order to have the lowest total cost that MacD would still find fair?

4.2: Today's LPP in summary form

- Variables:

U	=	Price to rent one acre of land this season
V	=	Price to control one dollar of MacD's capital
W	=	Price to use one hour of MacD's labor force

- Constraints:

  (price you offer for the crop A package needs to be at least what MacD would get from just planting it)

$$U + 50V + 20W \geq 150 \qquad \text{Crop A}$$
$$U + 60V + 25W \geq 200 \qquad \text{Crop B}$$
$$U \geq 0, V \geq 0, W \geq 0$$

- Objective:

  Minimize total cost $C = 100U + 6000V + 2400W$

4.2: Convert to equations

- $U + 50V + 20W \geq 150$ . . . could still add something to 150

$$150 + X = U + 50V + 20W$$

- $U + 60V + 25W \geq 200$ becomes

$$200 + Y = U + 60V + 25W$$

- X is how much higher the price for crop A would have to be to make selling the resources a bad idea for MacD
- Y is (ditto) for crop B
- Move the variables to the left, and numbers to the right:

$$\begin{array}{rrrrrl}
-U - & 50V - & 20W + X = & -150 & \text{Crop A} \\
-U - & 60V - & 25W + Y = & -200 & \text{Crop B} \\
\hline
-100U - & 6000V - & 2400W + C = & 0 & \text{Cost}
\end{array}$$

4.2: Convert to matrix

$$\begin{array}{rrrrrl} -U - & 50V - & 20W + X = & -150 & & \text{Crop A} \\ -U - & 60V - & 25W + Y = & -200 & & \text{Crop B} \\ \hline -100U - & 6000V - & 2400W + C = & 0 & & \text{Cost} \end{array}$$

- As a matrix:

$$\begin{array}{c} \begin{array}{cccccc} U & V & W & X & Y & C \end{array} \\ \left(\begin{array}{cccccc|c} -1 & -50 & -20 & \textcircled{1} & 0 & 0 & -150 \\ -1 & -60 & -25 & 0 & \textcircled{1} & 0 & -200 \\ \hline -100 & -6000 & -2400 & 0 & 0 & \textcircled{1} & 0 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Crop A} \\ \text{Crop B} \\ \text{Cost} \end{array}$$

- Still in RREF. Pivots are X, Y, C. U,V,W are free.
- Should we set the free variables to 0? That is a **seriously** low cost.

4.2: Our basic idea is not feasible

$$\begin{array}{cccccc|c} U & V & W & X & Y & C & RHS \\ \hline -1 & -50 & -20 & \textcircled{1} & 0 & 0 & -150 \\ -1 & -60 & -25 & 0 & \textcircled{1} & 0 & -200 \\ \hline -100 & -6000 & -2400 & 0 & 0 & \textcircled{1} & 0 \end{array} \quad \begin{array}{l} \\ \text{Crop A} \\ \text{Crop B} \\ \text{Cost} \end{array}$$

- If we set $U = V = W = 0$,
  then we get $X = -150$ and $Y = -200$
- So if we offer MacDonald \$0 for everything then he would be willing to sell even if the price of crop A rose by $-\$150 \ldots$
- MacD won't sell us at these rates until the price of crop A is \$0
- This solution is not feasible.
- It certainly is the lowest cost for those free variables
- Again, we have the wrong pivots!

4.2: Pivoting

- If you'd like you could learn to pivot these problems.
- You do everything sideways.

  Instead of column then row, you do row then column. Column ratios instead of row ratios. Still use row ops.
- Doing our work sideways is confusing.
- Instead we just turn the problem sideways!
- Bottom-line, how much would MacD sell out for this season?
- MacD could make \$19,200 if he used his resources (wisely) and so he would sell out if MacFrugal offered him (enough) more than \$19,200
- MacFrugal and MacDonald are solving the same problem!

4.2: Compare MacDonald and MacFrugal

- MacDonald

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P \end{array} \\ \left(\begin{array}{cccccc|c} 1 & 1 & \textcircled{1} & 0 & 0 & 0 & 100 \\ 50 & 60 & 0 & \textcircled{1} & 0 & 0 & 6000 \\ 20 & 25 & 0 & 0 & \textcircled{1} & 0 & 2400 \\ \hline -150 & -200 & 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Labor} \\ \text{Profit} \end{array}$$

- MacFrugal

$$\begin{array}{c} \begin{array}{cccccc} U & V & W & X & Y & C \end{array} \\ \left(\begin{array}{cccccc|c} -1 & -50 & -20 & \textcircled{1} & 0 & 0 & -150 \\ -1 & -60 & -25 & 0 & \textcircled{1} & 0 & -200 \\ \hline -100 & -6000 & -2400 & 0 & 0 & \textcircled{1} & 0 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Crop A} \\ \text{Crop B} \\ \text{Cost} \end{array}$$

- If we ignore the $\textcircled{1}$ part, they are very similar

4.2: Compare MacDonald and MacFrugal

- MacDonald without the negatives

X	Y	RHS	
1	1	100	Land
50	60	6000	Capital
20	25	2400	Labor
150	200	0	Profit

- MacFrugal without the negatives

U	V	W	RHS	
1	50	20	150	Crop A
1	60	25	200	Crop B
100	6000	2400	0	Cost

- Rows became columns, columns became rows

4.2: Reading the dual answer

- MacFrugal can see the answer to his problem in the matrix for MacDonald's answer:

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P \end{array} \\ \left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Crop B} \\ \text{Profit} \end{array}$$

- MacDonald reads the rows

$$U = 4, V = 240, Y = 96, X = 0, W = 0, P = 19200$$

- MacFrugal reads the columns

$$X = 10, Y = 0, U = 0, V = 0, W = 8, C = 19200$$

4.1: Negatives that don't get fixed

- HWB3 has 2 problems that ask you to determine whether the problem has an answer or not
- You'd like to give the best answer
- Sometimes there is no best answer
- Sometimes the homework problem is a license to print money
- How do you tell? Well, pivot once first.
- When you try to pivot again, there may be too many negatives

4.1: License to print money

- Pivot column: any negative in the bottom row
- Pivot row: smallest non-negative row ratio
- What if all of the row ratios are negative?

X	Y	U	P	RHS
0	-1	(1)	0	10
(1)	-2	0	0	30
0	-5	0	(1)	50

- The pivot column is Y,
  the row ratios are $\frac{10}{-1} = -10$ and $\frac{30}{-2} = -15$
- No pivot! No limit on Y!

$$U = 10 + Y, X = 30 + 2Y, P = 50 + 5Y$$

Ch 4: Drill

X	Y	U	P	RHS
0	-1	①	0	10
①	-2	0	0	30
0	5	0	①	50

Ch 4: Drill

X	Y	U	P	RHS
0	-1	①	0	10
①	-2	0	0	30
0	5	0	①	50

- This one is DONE.
  Bottom row is all not-negative

Ch 4: Drill

X	Y	U	P	RHS
0	-1	①	0	10
①	-2	0	0	30
0	5	0	①	50

- This one is DONE.
  Bottom row is all not-negative
- MacDonald's primal/standard solution: (Y is free)

$$U = 10, X = 30, Y = 0, P = 50$$

Ch 4: Drill

X	Y	U	P	RHS
0	-1	①	0	10
①	-2	0	0	30
0	5	0	①	50

- This one is DONE.
  Bottom row is all not-negative
- MacDonald's primal/standard solution: (Y is free)

$$U = 10, X = 30, Y = 0, P = 50$$

- MacFrugal's dual/minimization solution: (X,U are zero)

$$X = 0, Y = 5, U = 0, C = 50$$

Ch 4: Drill

X	Y	U	P	RHS
0	1	①	0	10
①	2	0	0	30
0	-5	0	①	50

Ch 4: Drill

X	Y	U	P	RHS
0	1	①	0	10
①	2	0	0	30
0	-5	0	①	50

- This one is not done!
- Only possible pivot column is Y
- Row ratios are $10/1$ and $30/2$
- Pivot row is first row, so new pivot is the 1
- Leave row 1, Row 2 minus 2 copies of Row 1, Row 3 plus 5 copies of Row 1

$$R_1; R_2 - 2R_1; R_3 + 5R_1$$

Ch 4: Drill

X	Y	U	P	RHS
0	0	①	0	10
①	2	0	0	30
0	-5	0	①	50

Ch 4: Drill

X	Y	U	P	RHS
0	0	①	0	10
①	2	0	0	30
0	-5	0	①	50

- This one is not done!
- Only possible pivot column is Y
- Row ratios are $10/0$ (so ignore!) and $30/2$
- Pivot row is second row, so new pivot is the 2
- Leave row 1, Row 2 divided by 2, Row 3 plus 5 copies of new Row 2

$$R_1; R_2/2; R_3 + 5R_{\text{new } 2}$$

Ch 4: Drill

X	Y	U	P	RHS
0	1	①	0	0
①	2	0	0	30
0	-5	0	①	50

Ch 4: Drill

X	Y	U	P	RHS
0	1	①	0	0
①	2	0	0	30
0	−5	0	①	50

- This one is not done!
- Only possible pivot column is Y
- Row ratios are $0/1$ and $30/2$; smallest non-negative is $0/1 = 0$!
- Pivot row is first row, so new pivot is the 1
- Leave row 1, Row 2 minus 2 copies of Row 1, Row 3 plus 5 copies of Row 1

$$R_1; R_2 - 2R_1; R_3 + 5R_1$$

Ch 4: Drill

X	Y	U	P	RHS
0	-1	①	0	10
①	2	0	0	30
0	-5	0	①	50

Ch 4: Drill

X	Y	U	P	RHS
0	-1	(1)	0	10
(1)	2	0	0	30
0	-5	0	(1)	50

- This one is not done!
- Only possible pivot column is Y
- Row ratios are $\frac{10}{-1}$ (negative, so ignore!) and $\frac{30}{2} = 15$
- Pivot row is second row, so new pivot is the 2
- Leave row 1, Row 2 divided by 2, Row 3 plus 5 copies of new Row 2

$$R_1; R_2/2; R_3 + 5R_{\text{new 2}}$$

Ch 4: Drill

X	Y	U	P	RHS
0	-1	(1)	0	10
(1)	-2	0	0	30
0	-5	0	(1)	50

Ch 4: Drill

X	Y	U	P	RHS
0	-1	①	0	10
①	-2	0	0	30
0	-5	0	①	50

- This one is not done!
- Only possible pivot column is Y
- Row ratios are $\frac{10}{-1}$ (negative, so ignore!) and $\frac{30}{-2}$ (negative, so ignore!)
- There is no pivot row! Profit is not bounded!