MA162: Finite mathematics

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SCHEDULE:

- HW C1 is due Monday, Mar 28th, 2011.
- HW C2 is due Monday, Apr 4th, 2011.
- HW C3 is due Sunday, Apr 10th, 2011.
- Exam 3 is Monday, Apr 11th, 5:00pm-7:00pm.

Today we will cover 5.2: annuities.   We will be using calculators today.

Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money

    - Simple interest

    - Compound interest

    - Sinking funds

    - Amortized loans

- Chapter 6, Counting

    - Inclusion exclusion

    - Inclusion exclusion

    - Multiplication principle

    - Permutations and combinations

5.2: Annuities

- "Annuity" can refer to a wide variety of financial instruments, often associated with retirement

- In this class, it is simply a steady flow of cash into an interest bearing account

- For instance, if you put \$100 at the end of every month into your savings account, earning 1% interest, then

- After 1 month: you have the original \$100

- After 2 months you have:
  \$100.00 from the first deposit,
  \$ 0.08 interest on the first deposit, and
  <u>\$100.00</u> from the second deposit,
  \$200.08 **Total**

5.2: Watching it grow

- After three months, you have:
  $200.08 from last month,
  $ 0.17 interest on last month's balance, and
  <u>$100.00 from the third deposit,</u>
  $300.25 **Total**

- After four months, you have:
  $300.25 from last month,
  $ 0.25 interest on last month's balance, and
  <u>$100.00 from the fourth deposit,</u>
  $400.50 **Total**

- How about after a year? Two years? Fairly tedious this way.

5.2: Formula

$$A = R((1 + i)^n - 1)/i$$

- where the **Recurring payment** is how much is deposited at the end of each period, like \$100

- the **interest rate** per period, like $1\%/12$

- the **number of periods**, like four months

- the **accumulated amount**, like

$$A = \$100((1 + 1\%/12)^4 - 1)/(1\%/12) = \$400.50$$

$$A = 100 \star ((1 + 0.01/12) \wedge 4 - 1)/(0.01/12) = 400.5001200$$

5.2: Examples of formula

$$A = R((1+i)^n - 1)/i$$

- After one year of investing \$100 at the end of every month at a 1% (nominal yearly) interest rate:
  - R = \$100
  - i = 1%/12
     ≈ 0.00833333
  - n = 12 months
  - A = \$100$((1 + 1\%/12)^{12} - 1)/(1\%/12) \approx$ \$1205.52

- After two years of investing \$100 at the end of every month at a 1% (nominal yearly) interest rate:
  - R = \$100
  - i = 1%/12
     ≈ 0.00833333
  - n = 24 months
  - A = \$100$((1 + 1\%/12)^{24} - 1)/(1\%/12) \approx$ \$2423.14

5.2: Retirement example

- UK employees aged 30 or over must contribute 5% of their salary each month to a retirement plan, which UK doubles, a total of 15%

- If a UK employee makes \$35k and retires at age 65 and manages to earn a steady 8% interest rate, then they retire with:

  $R = (\$35000)(15\%)/12 = \$437.50$

  $i = 8\%/12$

  $n = (35)(12) = 420$ months

  $A = \$437.50((1 + 8\%/12)^{420} - 1)/(8\%/12) \approx \$1,003,573.59$

- If a UK employee makes \$70k and retires at age 65 and manages to earn a steady 8% interest rate, then they retire with:

  $R = \$875$

  $i = 8\%/12$

  $n = (35)(12) = 420$ months

  $A = \$875((1 + 8\%/12)^{420} - 1)/(8\%/12) \approx \$2,007,147.18$

5.2: Sinking fund example

- Businesses can often predict future expenses; our building needs a new water boiler ($80k) after this one breaks

- We set aside a little each month so that we have it when we need it

- If we can get 3% interest in low-risk investments and expect the boiler to fail in 5 years, we need to invest R per month:

$$A = R((1 + i)^n - 1)/(i)$$

R	$= ?$
i	$= 3\%/12$
n	$= (5)(12) = 60$ months
A	$= \$80000$
$\$80000$	$= R((1 + 3\%/12)^{60} - 1)/(3\%/12)$
$\$80000$	$= R(64.64671280)$
R	$= \$80000/64.64671280 = \$1237.50$

5.2: Sinking fund versus one-time-investment

- Maybe we don't want to pay a little each month

- Maybe we just want to invest a whole bunch now and cash in later

$$A = P(1+i)^n$$

$$P = ?$$

$$i = 3\%/12$$

$$n = (5)(12) = 60 \text{ months}$$

$$A = \$80000$$

$$\$8000 = P(1 + 3\%/12)^{60}$$

$$\$8000 = P(1.161616782)$$

$$P = \$80000/1.161616782 = \$68869.53$$

- Less total money we invested for same future value

- But we need that \$68k NOW, not \$1.2k at a time

5.2: Why does the formula work?

- After one month you have $100

- The next month you add a fresh $100 and (1+i) times your previous month
  $$\$100 + \$100 \cdot (1 + i)$$

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  $$\$100 + (\$100 + \$100 \cdot (1 + i)) \cdot (1 + i)$$
  $$\$100 + \$100 \cdot (1 + i) + \$100 \cdot (1 + i)^2$$

- The next month you add a fresh $100 and (1+i) times your previous month
  $$\$100 + (\$100 + (\$100 + \$100 \cdot (1 + i)) \cdot (1 + i)) \cdot (1 + i)$$
  $$\$100 + \$100 \cdot (1 + i) + \$100 \cdot (1 + i)^2 + \$100 \cdot (1 + i)^3$$

5.2: Trick for summations

- After n months you have added up n things:

$$A = \$100 + \$100 \cdot (1 + i) + \cdots + \$100 \cdot (1 + i)^{n-1}$$

- Let's try a trick. What happens if I let the money ride for a month? It earns interest, so I have $A \cdot (1 + i)$ in the bank.

- How much more is that? Well $A \cdot (1 + i) - A = Ai$ is not tricky.

- But multiply it out before doing the subtraction:

	$A \cdot (1 + i)$	$=$			$\$100 \cdot (1 + i)$	$+$	$\ldots$	$+$	$\$100 \cdot (1 + i)^{n-1}$	$+$	$\$100 \cdot (1 + i)^n$
$-$	A	$=$	$\$100$	$+$	$\$100 \cdot (1 + i)$	$+$	$\ldots$	$+$	$\$100 \cdot (1 + i)^{n-1}$		
	Ai	$=$	$-\$100$							$+$	$\$100 \cdot (1 + i)^n$

- So $Ai = \$100 \cdot ((1 + i)^n - 1)$ and we can solve for A:

$$A = \$100 \frac{(1 + i)^n - 1}{i}$$

5.2: Time value of money and total payout

- How much would you pay me for (the promise of) $100 in a year?

- Future money is not worth as much as money right now
  "A bird in the hand, is worth two in the bush" posits an interest rate of 100%

- Present value of future money **depreciates** the value of future money by comparing it to present money invested in the bank now

- **Total payout** is a popular measure of a financial instrument, but it mixes present money, with in-a-little-while money, with future money

- Total payout of an annuity is just the total amount you put in the savings account (or the total amount you borrowed each month)

5.2: Summary

- Today we learned about **annuities**, **present value**, **future value**, and **total payout**

  - Future value of annuity, paying out n times at per-period interest rate i
  $$A = R\frac{(1 + i)^n - 1}{i}$$

  - Present value of annuity is just future value divided by $(1 + i)^n$

  - Total payout is just nR, n payments of R each

- You are now ready to complete HWC1 (and should have done 3 or 4 problems)

- Make sure to take advantage of office hours

5.3: Buying annuities

- How much would you pay today for an annuity paying you back $100 per month for 12 months?

- No more than $1200 for sure,
  if you had $1200 you could just pay yourself

- Let's try to find the right price for such a cash flow

- What if you didn't need the money?
  You could deposit it each month into your savings account.

- We already calculated that you end up with $1205.52 if you do that

- How much would you pay today for $1205.52 in the bank a year from now?

5.3: Pricing annuities

- If you had \$1193.53 and just put it in the bank now, you'd end up with $\$1193.53(1 + 1\%/12)^{12} = \$1205.52$ anyways

- If you were just concerned with how much you had in the bank at the end, then you would have no preference between \$1193.53 up front and \$100 each month.

- In other words, the **present value** of the \$100 each month for a year is \$1193.53 because both of those have the same **future value**

- What if you do need the money each month?
  Is \$1193.53 still the right price?

5.3: Pricing annuities again

- What would happen if you put $1193.53 in the bank, and withdrew $100 each month?

- At the end of the year, you'd have $0.00 in the bank, but you would not be overdrawn.

- Why is that? Imagine borrowing money from your friend, $100 every month and not paying them back

- They know you pretty well, so they insisted on 1% interest, compounded monthly

- How much do you owe them at the end?

- Well from their point of view, they gave their money to you, just like putting it in a savings account

- The bank would have owed them $1205.52, so you owe them $1205.52. Now imagine your savings account is your friend.