MA111: Contemporary mathematics

Jack Schmidt

University of Kentucky

October 12, 2011

SCHEDULE:

- HW Ch 5 Part One is due Today, Oct 12th.
- HW Ch 5 Part Two is due Wed, Oct 19th.
- Exam 3 is Monday, Oct 24th, during class.
- Exams not graded yet (and this week is busy; will be done for midterms)

Today we will recognize some patterns

5.5: Handshaking theorem

- 5 people are in a room, and they shake hands with some people (but not themselves)
- They shook the following number of hands:
  3, 2, 3, 3, 1

5.5: Handshaking theorem

- 5 people are in a room, and they shake hands with some people (but not themselves)
- They shook the following number of hands:
  3, 2, 3, 3, 1
- How many total? $3 + 2 + 3 + 3 + 1 = 12$

5.5: Handshaking theorem

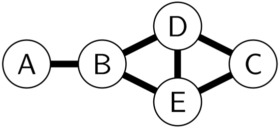
- 5 people are in a room, and they shake hands with some people (but not themselves)
- They shook the following number of hands:
  3, 2, 3, 3, 1
- How many total? $3 + 2 + 3 + 3 + 1 = 12$
- How many handshakes total?

5.5: Handshaking theorem

- 5 people are in a room, and they shake hands with some people (but not themselves)
- They shook the following number of hands:
  3, 2, 3, 3, 1
- How many total? $3 + 2 + 3 + 3 + 1 = 12$
- How many handshakes total?
- Draw a graph representing their handshakes.

5.5: Handshaking theorem

- 5 people are in a room, and they shake hands with some people (but not themselves)
- They shook the following number of hands:
  3, 2, 3, 3, 1
- How many total? $3 + 2 + 3 + 3 + 1 = 12$
- How many handshakes total?
- Draw a graph representing their handshakes.
- Only one possibility:



5.5: Another reconstruction

- 6 people, everyone shook hands with 2 people

5.5: Another reconstruction

- 6 people, everyone shook hands with 2 people
- How many total? $2+2+2+2+2+2=12$
  How many actual handshakes?

5.5: Another reconstruction

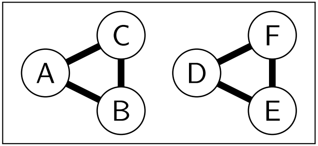
- 6 people, everyone shook hands with 2 people
- How many total? $2+2+2+2+2+2=12$
  How many actual handshakes?
- Draw the graph.

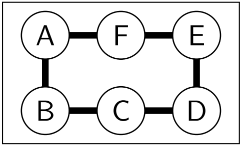
5.5: Another reconstruction

- 6 people, everyone shook hands with 2 people
- How many total? $2+2+2+2+2+2=12$
  How many actual handshakes?
- Draw the graph.
- Two possibilities this time.

5.5: Another reconstruction

- 6 people, everyone shook hands with 2 people
- How many total? $2 + 2 + 2 + 2 + 2 + 2 = 12$
  How many actual handshakes?
- Draw the graph.
- Two possibilities this time.
- They are:





5.5: Another reconstruction

- 4 people shook hands: 1, 3, 2, 2

5.5: Another reconstruction

- 4 people shook hands: 1, 3, 2, 2

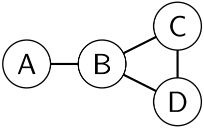
- Total is $1 + 3 + 2 + 2 = 8$, but really only 4 handshakes (double counted)

5.5: Another reconstruction

- 4 people shook hands: 1, 3, 2, 2
- Total is $1 + 3 + 2 + 2 = 8$, but really only 4 handshakes (double counted)
- Draw it!

5.5: Another reconstruction

- 4 people shook hands: 1, 3, 2, 2
- Total is $1 + 3 + 2 + 2 = 8$, but really only 4 handshakes (double counted)
- Draw it!
- Only one possibility:



5.5: Another reconstruction

- 5 people, each shook one other person's hand
- How many total?
- Draw it.

5.5: Another reconstruction

- 5 people, each shook one other person's hand
- How many total?
- Draw it.
- How many actual handshakes happened?

5.5: Another reconstruction

- 5 people, each shook one other person's hand
- How many total?
- Draw it.
- How many actual handshakes happened?
- So make a statement.
  What has to be true about the number of handshakes in a handshaking graph?

5.5: Our first theorem

Theorem (Handshaking theorem)

The total degree of a graph is equal to twice the number of edges.

Proof.

□

5.5: Our first theorem

Theorem (Handshaking theorem)

The total degree of a graph is equal to twice the number of edges.

Proof.

Each edge counts twice for the total degree: once for its start and once for its end. Hence the total degree is at least twice the number of edges. All parts of the degree come from counting edges, so the total degree is no more than twice the number of edges. Hence they are equal. □

5.5: Our first theorem

Theorem (Handshaking theorem)

The total degree of a graph is equal to twice the number of edges.

Proof.

Each edge counts twice for the total degree: once for its start and once for its end. Hence the total degree is at least twice the number of edges. All parts of the degree come from counting edges, so the total degree is no more than twice the number of edges. Hence they are equal. □

Corollary

There is no graph of total degree 5. There is no graph with only 5 vertices, all of degree 1.

5.5: Our first converse

- Our corollary told us that some graphs were impossible.
- If we don't allow people to shake hands with themselves, and only allow a pair to shake hands or not (rather than say shake twice) then there are more restrictions.
- 2 people: each shakes hands with 2. $2 + 2 = 4$ is even, but...
- If we allow loops and multiple edges though, it is easy:

Theorem

For every sequence of non-negative integers that totals to an even number, there is a graph (possibly with loops and multiple edges) with that degree sequence.

5.5: Another reconstruction

- Draw a graph with vertices of degrees: 10, 2, 0, 3, 5
- Draw a graph with vertices of degrees: 2, 2, 2, 2, 2, 2
- Draw a handshaking graph with 6 people, each shaking hands with one person
- Draw a graph with vertices of degrees: 1, 2, 3, 4, 3
- Draw a graph with vertices of degrees: 0, 0, 0, 0, 0
- How many graphs with vertices of degrees: 3, 3, 4, 4, 4, 4
  (Quiz; get at least one)
- A hard way to do this is to check the encyclopedia.

5.5: Back to the Euler paths

- We want to prove a similar counting theorem about the existence of Euler paths and circuits
- Some graphs are traceable, some are not
- Some require you to start and end at a specific point
- We want to be able to tell quickly which is which

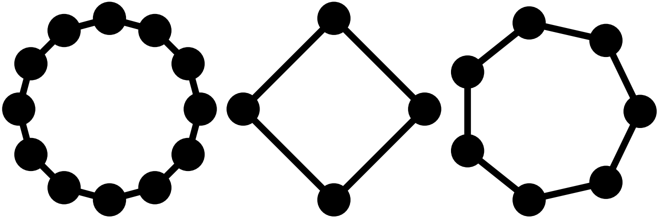
Lemma

The reverse of an Euler path is also an Euler path.
A shifted Euler circuit is still an Euler circuit.

In other words, if ABCDCAB is an Euler path, so is BACDCBA.
If ABCDCABA is an Euler circuit, then so is BCDCABAB and CDCABABC, etc.

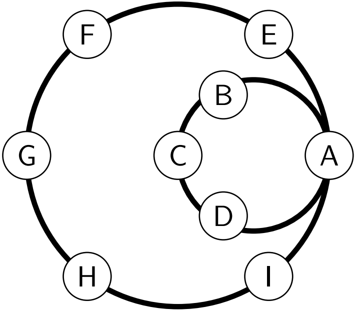
5.5: Changing the problem a little

- The lemma says "start" and "end" are interchangable for a path
- The lemma says we can start a circuit *anywhere*
- Circuits sound easier because we can start at any vertex, and choose any edge as the "first" edge.
- What do we know about the degrees of the vertices in a circuit?



5.5: Some circuits are more complicated

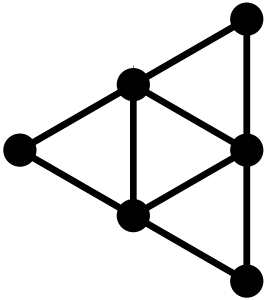
- The whole graph is not always just a circle



- But am I right? It is easy to find an Euler circuit.
- The degree of all the vertices on a single circle is 2.
- "A" is on two circles. Its degree is $2 + 2 = 4$.

5.5: Weird overlaps

- Sometimes the graph can be thought of as layers in more than one way



- Euler circuits can be thought of as layers of circles
- Draw this as three circuits (don't reuse edges)
- Draw this as two circuits (don't reuse edges)
- What is the degree of each vertex? How many circles is it on?

5.5: Ok, can we do the theorem?

- Is every vertex on a circle?

5.5: Ok, can we do the theorem?

- Is every vertex on a circle?
- We don't know how many circles, but what do we know about the degree?

5.5: Ok, can we do the theorem?

- Is every vertex on a circle?
- We don't know how many circles, but what do we know about the degree?
- Ok, so we have Euler's theorem:

Theorem (Euler, 1736)

If a graph has an Euler circuit, every vertex has even degree.

5.5: Ok, can we do the theorem?

- Is every vertex on a circle?
- We don't know how many circles, but what do we know about the degree?
- Ok, so we have Euler's theorem:

Theorem (Euler, 1736)

If a graph has an Euler circuit, every vertex has even degree.

- What about the converse? Is having even degree enough to give us an Euler circuit?

5.5: Ok, can we do the theorem?

- Is every vertex on a circle?
- We don't know how many circles, but what do we know about the degree?
- Ok, so we have Euler's theorem:

Theorem (Euler, 1736)

If a graph has an Euler circuit, every vertex has even degree.

- What about the converse? Is having even degree enough to give us an Euler circuit?
- Euler didn't prove it. It took a real Hierholzer to do it.

5.5: Ok, can we do the theorem?

- Is every vertex on a circle?
- We don't know how many circles, but what do we know about the degree?
- Ok, so we have Euler's theorem:

Theorem (Euler, 1736)

If a graph has an Euler circuit, every vertex has even degree.

- What about the converse? Is having even degree enough to give us an Euler circuit?
- Euler didn't prove it. It took a real Hierholzer to do it.
- Find a graph with all even degree, but no Euler circuit!

5.5: The full theorems

- First the Euler circuit:

Theorem (Hierholzer, 1873)

A graph has an Euler circuit if and only if it is connected and every vertex has even degree.

- Then the Euler path (just remove an edge from the circuit to get a path, or add an edge to a path to make it a circuit):

Corollary

A graph has an Euler path if and only if it is connected and exactly two of its vertices have odd degree.

- Also, there is a web game.