

MA111: Contemporary mathematics

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SCHEDULE:

- HW Ch 5 Part Two is due Wed, Oct 19th.
- Exam 3 is Monday, Oct 24th, during class.
- Exams not graded yet (and this week is busy; will be done for midterms)

Today we will recognize another pattern

5.5: Our first theorem

Theorem (Handshaking theorem)

The total degree of a graph is equal to twice the number of edges.

Corollary

There is no graph of total degree 5. There is no graph with only 5 vertices, all of degree 1.

Theorem (Converse)

For any sequence of non-negative integers whose sum is even, there is a (possibly loopy and multiple-edged) graph whose vertices have exactly those degrees.

5.5: Back to the Euler paths

- We want to prove a similar counting theorem about the existence of Euler paths and circuits
- Some graphs are traceable, some are not
- Some require you to start and end at a specific point
- We want to be able to tell quickly which is which

Lemma

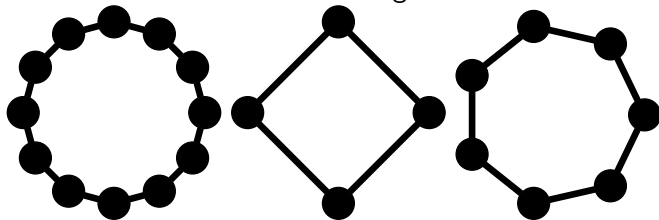
The reverse of an Euler path is also an Euler path.

A shifted Euler circuit is still an Euler circuit.

In other words, if ABCDCAB is an Euler path, so is BACDCBA. If ABCDCABA is an Euler circuit, then so is BCDCABAB and CDCABABC, etc.

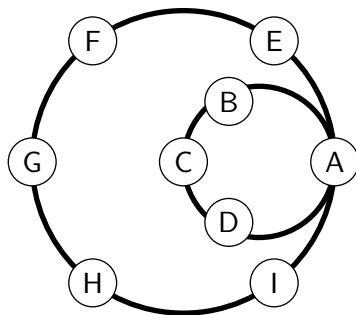
5.5: Changing the problem a little

- The lemma says “start” and “end” are interchangeable for a path
- The lemma says we can start a circuit *anywhere*
- Circuits sound easier because we can start at any vertex, and choose any edge as the “first” edge.
- What do we know about the degrees of the vertices in a circuit?



5.5: Some circuits are more complicated

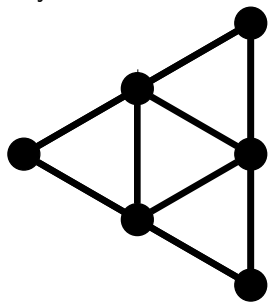
- The whole graph is not always just a circle



- But am I right? It is easy to find an Euler circuit.
- The degree of all the vertices on a single circle is 2.
- “A” is on two circles. Its degree is $2 + 2 = 4$.

5.5: Weird overlaps

- Sometimes the graph can be thought of as layers in more than one way



- Euler circuits can be thought of as layers of circles
- Draw this as three circuits (don't reuse edges)
- Draw this as two circuits (don't reuse edges)
- What is the degree of each vertex? How many circles is it on?

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If a graph has an Euler circuit, every vertex has even degree.

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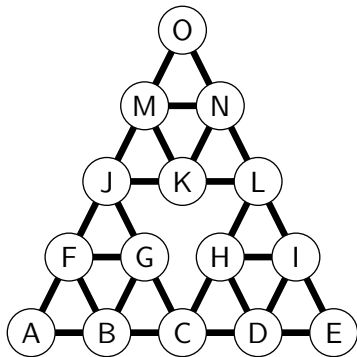
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- What about the converse? Is having even degree enough to give us an Euler circuit?
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- Find a graph with all even degree, but no Euler circuit!

5.5: Difficult, impossible, or just plain easy



5.5: The full theorems

- First the Euler circuit:

Theorem (Hierholzer, 1873)

A graph has an Euler circuit if and only if it is connected and every vertex has even degree.

- Then the Euler path (just remove an edge from the circuit to get a path, or add an edge to a path to make it a circuit):

Corollary

A graph has an Euler path if and only if it is connected and exactly two of its vertices have odd degree.

- Also, there is a [web game](#).