MA111: Contemporary mathematics

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Schedule:

- Final projects are due Nov 21, 2011.
- I expect to spend all day Thursday grading prelim projects (approximately 10 more hours of grading; 8 hour work day)
- Symmetry exam is looking good, thanks!
- Final exam is the next exam.

Today we will introduce games, utility functions, and fair division.

A game: the rules

- **Requirements:** Two players and 10 one-dollar bills
- **Rules:** The game proceeds in three steps:
  1. Flip a coin to determine the first player
  2. First players says a number from 1 to 10, n
  3. Second player says "Yes" or "No"
- **Outcome:** If the second player says "Yes", then second player takes n one-dollar bills, and the first player gets the rest.
  If the second player says "No", then the first player takes n one-dollar bills, and the second player gets the rest.

A game: what is the best strategy?

- Clearly the coin flip makes a big difference in strategy

  (if your strategy is "Say Yes!!!" but you are the first player, you'll just be disqualified)

- The first player's strategy is a little hard to analyze

- The second player's strategy is easy: what strategy maximizes the money?

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- Say "Yes" if $n > 5$, say "No" if $n < 5$. Say either if $n = 5$.

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- First player can figure out the second players strategy!

  If $n > 5$, then first player gets stuck with the small "half".
  If $n < 5$, then first player gets stuck with the small "half".

  What should the first player do???

A game: the strategy

- If first player, then say “5”
- If second player, then answer the question “Is $n > 5$?”

- As first player, you always get \$5
- As second player, you always get at least \$5

The beginning of a fairy tale

A brother and sister were walking through the woods, when they saw a funny little man sitting on a stump with a twenty dollar bill. They shoved each other out of the way to race up to the funny little man and ask him for the twenty dollar bill. The funny little man introduced himself as the evil imp Sweeny. "In truth, I have no use for this twenty dollar bill, but I see the two of you have many one dollar bills and I would be willing to trade." The two children pushed and shoved each other trying to give the imp a dollar bill. "This will not do. We shall play a game to see who gets the twenty. Each will take a turn saying Raise or Fold, and you will give me that many one-dollar bills for the twenty." The brother smirked and said "I have more one dollar bills than she does, so I will surely win!" The sister growled and said "Well, I will raise until I have no more dollars, so it will be a very costly twenty for you!" The imp raised his funny little hands and said, "I haven't finished the rules: to make sure each one is bidding honestly, I require BOTH of you to pay your last bid, even though only the highest bid will get the twenty."
What should the two children do?

An evil game: the rules

- **Requirements:**
  Two players with their own money;
  a 20-dollar bill;
  an evil imp named Sweeney

- **Rules:** The game proceeds in three stages:

  1. Flip a coin to choose the first bidder
  2. Take turns saying either
     - "Raise" and go back to step 2, or
     - "Fold" to move to stage 3
  3. Suppose the players said "Raise" exactly n times
     If no one said "Raise", then the imp keeps his twenty and leaves
     The last player to say "Raise" pays the imp $\$n$ and takes the twenty
     The other player pays the imp $\$(n-1)$ dollars

An evil game: what is the best strategy?

- Neither player's strategy is clear, so maybe they should just do what seems best at the time?
- The first player has a choice:
  Raise to get \$20 for \$1, or
  Fold to get \$0 for \$0.
  Which is better?

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- The second player has a choice:
  Raise to get \$20 for \$2, or
  Fold to get \$0 for \$0.

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  Raise to get \$20 for \$2, or
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- The first player has a choice:
  Raise to get \$20 for \$3, or
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  Raise to get $20 for $2, or
  Fold to get $0 for $0.
- The first player has a choice:
  Raise to get $20 for $3, or
  Fold to get $0 for $1.
- The second player has a choice:
  Raise to get $20 for $4, or
  Fold to get $0 for $2.

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- The second player has a choice:
  Raise to get \$20 for \$4, or
  Fold to get \$0 for \$2.
- What happens if they keep thinking like this?

An evil game: the only winning strategy

- Unless the children work together, they will just bid until they run out of money, because after the first turn it is always $18 better to say “Raise” than “Fold”
- The only way to win is not to play
- Of course if the children work together, they can get twenty for one, but this is a Grim fairy tale, and children normally end up in terrible circumstances, so I wouldn’t count on them cooperating.
- This fairy tale actually happens about once per month on cable TV with 8 to 10 million viewers.

Utility functions: how much is that worth to you?

- In these two games the prize was cash money; its value is "objective"
- What if the prize was a bar of chocolate?
- Perhaps the brother is a chocolate fiend, but the sister just ate.
- Perhaps the brother values the bar equally with twenty dollars
  The sister values the bar equally with 50 cents
- The sister would always fold, so a very different game.

Fair division: what is fair?

- When people value an item differently, how can we split it fairly?
- If it is 10 dollars and two people, then 5 dollars each is “fair”
- If it is a chocolate bar and 10 dollars, then it is not so clear...

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- Brother thinks he got 10 dollars, and she got 20 dollars. NOT FAIR.

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- Brother thinks he got 10 dollars, and she got 20 dollars. NOT FAIR.
- Sister thinks she got 50 cents, and he got 10 dollars. NOT FAIR.