MA111: Contemporary mathematics

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April 11, 2012

Schedule:

- Homework for today is a worksheet due at the beginning of class on Friday
- Written project due Friday April 20th: Two well-written homework answers:
  1. Answer one of #77, #78, #79, #80
  2. Answer one of #81, #82, #83

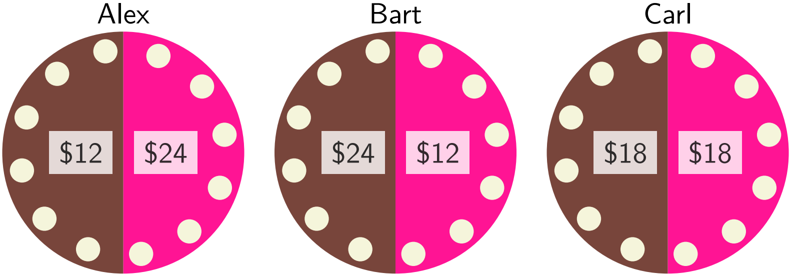
Today we will study how three players may fight over a division.

The loot

- A half-chocolate and half-strawberry cake has been purchased by Alex, Bart, and Carl, each paying $12.

The loot in the eye of the beholder

- There is only one cake, but there are three cake-a-vores, each with their own ideas of what the cake is worth:



- Alex likes Strawberry twice as much as Chocolate, Bart likes Chocolate twice as much as Strawberry, and Carl likes Chocolate and Strawberry equally.

A fair and perfect division: divide the values too

- We now assume the cakery already cut the cake into thirds:

A fair and perfect division: mixed piece as single piece

- How does each person value the thirds?

A fair and perfect division: who gets what?

- Which piece should each person get?
- There are six possible assignments

  (A gets one of three, B gets one of the other two, and C gets the rest, $(3)(2)(1) = 6$)
- One assignment seems most reasonable:

- We just gave Alex and Bart what they wanted, Carl didn't care

A fair and perfect division: is it a good one?

- Which piece should each person get?

- It is **fair**: each person paid $12, and got at least $12 back
- It is **envy-free**: no person actively wants to trade pieces
- It is **Pareto-optimal**: no person actively wants to trade pieces with people willing to trade
- It is NOT **equitable**: Alex and Bart are happier than Carl

A more perfect division: Able to Equit

- Carl complains the division is not equitable, and a fight erupts
- Before they do any damage, the evil imp Sweeny appears and offers to divide the cake equitably

"Give piece a chance!"
– Sweeny

- $\frac{3}{4}$ of the Chocolate is one piece, $\frac{3}{4}$ of the Strawberry is one piece, and $\frac{1}{4}$ of each is the last piece

A more perfect division: Values

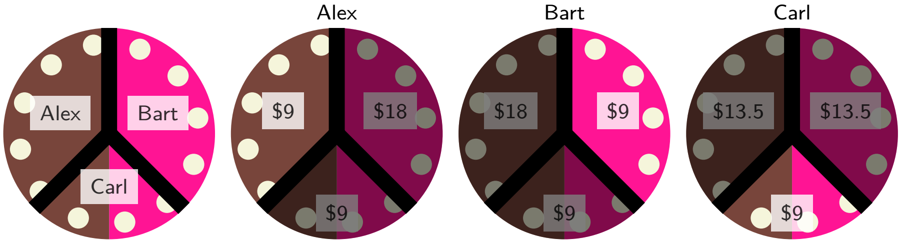
- Alex, Bart, and Carl estimate the value of the pieces:

A more perfect division: Values

- Alex, Bart, and Carl estimate the value of the pieces:

A more perfect division: Equitable assignment

- The evil imp Sweeny makes the assignment of pieces:



- Alex, Bart, and Carl cringe (equally)
- This is NOT **fair**: Each paid $12, but received $9 of cake
- This is NOT **envy-free**: Everyone wants to trade with someone
- This is NOT **Pareto optimal**: Everyone wants to trade with someone willing to trade
- This is **equitable**: Everyone is equally (un)happy

A more perfect division: Self assignment

- The evil imp Sweeny is not done yet. ``Oh, I'm sorry, maybe equitability is not so desirable. Perhaps you should assign the pieces yourself!''
- Carl wants to trade with either Alex or Bart
  (who are willing but not thrilled to trade their \$9 piece for Carl's \$9 piece)
- Alex wants to trade with Bart, and Bart wants to trade with Alex
- After trading until no group can reach a trade consensus (Pareto-optimality) we have one of the following three situations (depending on whether Carl managed to trade early or not)
- Of the six assignments: only the worst is equitable, none are fair, none are envy-free.

A more perfect division: Three unhappy endings

- Each division is Pareto-optimal but unfair:

Alex and Bart trade
Carl suffers

Carl and Bart trade
Alex and Carl trade
Bart suffers

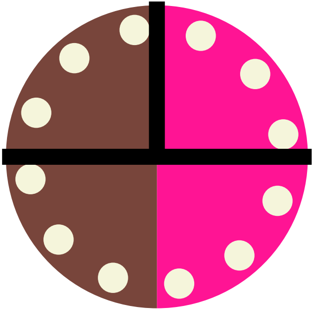
Carl and Alex trade
Bart and Carl trade
Alex suffers

What went wrong? Was it weird or evil?

- One theory is that one should not talk to evil imps
- But really, the first division was "even": three people, thirds
- The second division was lop-sided and weird
- Maybe it wasn't evil so much as just weird

A lumpy division can be fair

- I think some people underestimate Sweeny,
  so here is my suggestion:



"I know which piece I want!"
– Sweeny

- $\frac{1}{2}$ of the Chocolate is one piece, $\frac{1}{2}$ of the Strawberry is one piece, and $\frac{1}{2}$ of each is the last piece

A lumpy division can be fair: Values

- Alex, Bart, and Carl calculate their values:

A lumpy division can be fair: Values

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A lumpy division can be fair: Values

- Evil Dr. Jack suggests Carl gets first pick:

- This is fair and Pareto optimal, but not equitable or envy-free

- Both Alex and Bart want to trade with Carl,
  but Carl is not willing to trade with Alex or Bart

Added value: Survey of past divisions

- One thirds: \$16+\$16+\$12 = \$44
- Sweeny: \$9+\$9+\$9 = \$27
- Traded: \$18+\$18+\$9 = \$45
- Jack: \$12+\$12+\$18 = \$42
- Wide range of total value, paid \$36 got \$27 to paid \$36 got \$45
- What is the maximum total value?

Added value: Maximizing the value

- Intuitive: give the cake to whoever values it most
- All the Strawberry to Alex for \$24 and
  all the Chocolate to Bart for \$24
- \$48 total is the highest!
- In order to maximize happiness in the community,
  We should give all the wealth to the greedy people,
  and leave the moderates with nothing!

"The good of the many outweighs the good of the few."
– Sweeny
"Or the one"
– Carl

In MA162 we learn to solve these problems (without calculus):

$$\text{Maximize A+B+C subject to: } \left\{ \begin{array}{c} A = 12c_a + 24s_a \\ B = 24c_b + 12s_b \\ C = 18c_c + 18s_c \\ 1 = c_a + c_b + c_c \\ 1 = s_a + s_b + s_c \end{array} \right\} \text{ and } c_a, c_b, c_c, s_a, s_b, s_c \geq 0$$

Unique solution is $A = 24, B = 24, C = 0, c_a = c_c = s_b = s_c = 0, s_a = c_b = 1$

Added value: Can't we be equitable?

- If we listen to Sweeny, then we'll get the idea that equitable sharing is no good for anyone.

- However, if we ask our friends in MA162, they can find us a much better solution (assuming they can convince Alex, Bart, and Carl to reveal their true feelings about cake)

$$\text{Maximize } A + B + C \text{ subject to } \left\{ \begin{array}{c} A = 12c_a + 24s_a \\ B = 24c_b + 12s_b \\ C = 18c_c + 18s_c \\ 1 = c_a + c_b + c_c \\ 1 = s_a + s_b + s_c \\ A = B = C \end{array} \right\} \text{ and } c_a, c_b, c_c, s_a, s_b, s_c \geq 0$$

Unique solution is $A = B = C = \$14.40$, $c_a = s_b = 0$, $c_b = s_a = \frac{3}{5}$, $c_c = s_c = \frac{2}{5}$

- Everyone can get \$14.40 worth of cake if we give $\frac{3}{5}$ of the Strawberry to Alex, $\frac{3}{5}$ of the Chocolate to Bart, and the rest to Carl. This \$43.20 total, not too shabby.

Good news

- So we've seen some good news:
- We can maximize the total happiness, but at a cost to individuals
- We can mazimize an equitable happiness, but only with psychic mathies
- But there is more good news:
- Alex, Bart, and Carl can find themselves a fair share with no outside interference!
- There are simple games with clear rules to divide the loot
- There is a simple strategy to guarantee a fair share, even against an army of sociopathic competitors