MA162: Finite mathematics

Jack Schmidt

University of Kentucky

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Schedule:

- HW 3.2, 3.3 due Friday Feb 24, 2012
- HW 4.1 due Friday Mar 2, 2012
- Exam 2 is Monday, Mar 5, 2012 from 5pm to 7pm in CB106 and CB118

Today we will cover 4.1: Simplex algorithm

Exam 2: Overview

- 22% Ch. 2, Matrix arithmetic
- 33% Ch. 3, Linear optimization with 2 variables
  1. Graphing linear inequalities
  2. Setting up linear programming problems
  3. Method of corners to find optimum values of linear objectives
- 45% Ch. 4, Linear optimization with millions of variables
  1. Slack variables give us flexibility in RREF
  2. Some RREFs are better (business decisions) than others
  3. Simplex algorithm to find the best one using row ops
  4. Accountants and entrepreneurs are two sides of the same coin

4.1: Linear programming problems

- An LPP has three parts:
  - The variables (the business decision to be made)
  - The inequalities (the laws, constraints, rules, and regulations)
  - The objective (maximize profit, minimize cost)
- If there are more than two variables, graphing gets hard!
- We need to use matrices to handle so many variables.

4.1: Today's LPP as word problem

- Old MacDonald had 100 acres, \$6000, and 2400 labor hours
- Crop A costs him \$50/acre and 20hrs/acre in labor
- Crop B costs him \$60/acre and 25hrs/acre in labor
- Crop A earns him \$150/acre and Crop B earns him \$200/acre
- How many acres of each crop should he plant?

4.1: Today's LPP in summary form

- Variables:

  X = Number of acres of crop A to plant
  Y = Number of acres of crop B to plant

- Constraints:

$$\begin{aligned} X + Y &\leq 100 && \text{Land} \\ 50X + 60Y &\leq 6000 && \text{Capital} \\ 20X + 25Y &\leq 2400 && \text{Labor} \end{aligned}$$

- Objective:

  Maximize $P = 150X + 200Y$

4.1: Inequalities are just equalities in disguise

- All of our variables are non-negative (realistic)
- Think about $X + Y \leq 100$
- It means $X + Y$ has not yet exceeded 100
- We could still add something to bump it up to 100
- $X + Y + U = 100$ with $X, Y, U \geq 0$
- U is "the rest" of the 100, the "unused" land, the **slack**

$$U = 100 - (X + Y)$$

- "$U = 20$" means we left 20 acres fallow (unused)

4.1: LPP are just systems of equations in disguise

$$\begin{aligned} X + Y &\leq 100 && \text{Land} \\ 50X + 60Y &\leq 6000 && \text{Capital} \\ 20X + 25Y &\leq 2400 && \text{Labor} \end{aligned}$$

- Define our slack variables:

$$\begin{aligned} U &= 100 - (X + Y) && \text{unused Land} \\ V &= 6000 - (50X + 60Y) && \text{unused Capital} \\ W &= 2400 - (20X + 25Y) && \text{unused Labor} \end{aligned}$$

- Even profit is an equation: $P = 150X + 200Y$ just means $-150X - 200Y + P = 0$
- Now we have a system of equations:

$$\begin{aligned} X + Y + U &= 100 && \text{Land} \\ 50X + 60Y + V &= 6000 && \text{Capital} \\ 20X + 25Y + W &= 2400 && \text{Labor} \\ \hline -150X - 200Y + P &= 0 && \text{Profit} \end{aligned}$$

4.1: Write it as a matrix

- The system of equations:

$$\begin{array}{rrrrrrl}
X & + & Y & + & U & = 100 & \text{Land} \\
50X & + & 60Y & + & V & = 6000 & \text{Capital} \\
20X & + & 25Y & + & W & = 2400 & \text{Labor} \\
\hline
-150X & - & 200Y & + & P & = 0 & \text{Profit}
\end{array}$$

- Now as a matrix:

$$\begin{array}{c}
\begin{array}{cccccc}
X & Y & U & V & W & P
\end{array} \\
\left(\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 0 & 0 & 100 \\
50 & 60 & 0 & 1 & 0 & 0 & 6000 \\
20 & 25 & 0 & 0 & 1 & 0 & 2400 \\
\hline
-150 & -200 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
\end{array}
\begin{array}{l}
\\
\text{Land} \\
\text{Capital} \\
\text{Labor} \\
\\
\text{Profit}
\end{array}$$

4.1: Analyze the matrix

- Our matrix is basically in RREF!

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P & RHS \end{array} \\ \left(\begin{array}{cccccc|c} 1 & 1 & \textcircled{1} & 0 & 0 & 0 & 100 \\ 50 & 60 & 0 & \textcircled{1} & 0 & 0 & 6000 \\ 20 & 25 & 0 & 0 & \textcircled{1} & 0 & 2400 \\ \hline -150 & -200 & 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Labor} \\ \text{Profit} \end{array}$$

- U,V,W,P have pivots, and X,Y are free.
  We can make X and Y whatever we want!

- Wait. What do we want them to be?
  Isn't that why we started doing this anyways?

- OMG We have the wrong pivots!

4.1: Operation Row Op to the rescue!

- Well, what if we decided the 25 should have been a pivot?
- Easy to fix using row ops:

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ 50 & 60 & 0 & 1 & 0 & 0 & 6000 \\ 20 & 25 & 0 & 0 & 1 & 0 & 2400 \\ \hline -150 & -200 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_4+8R_3 \\ R_3/25}]{\substack{R_1-0.04R_3 \\ R_2-2.4R_3}} \left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array}\right)$$

- Y, U, V, P are pivots, X and W are free.

4.1: What does it say now?

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P \end{array} \\ \left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Crop B} \\ \text{Profit} \end{array}$$

- First row says $U = 4 - (1/5)X + (1/25)W$
- Last row says $P = 19200 - 10X - 8W$
- X and W are free, what should they be?
- Every acre of crop A we plant costs us \$10!

4.1: Summary

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P \end{array} \\ \left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Crop B} \\ \text{Profit} \end{array}$$

- Our free variables are $X = 0$ and $W = 0$
  Plant no acres of crop A, and use all available labor
- First row says $U = 4 - (1/5)X + (1/25)W = 4$
  Leave 4 acres fallow
- Second row says $V = 240 - 2X + (12/5)W = 240$
  Leave $240 unspent
- Third row says $Y = 96 - (4/5)X - (1/25)W = 96$
  Plant 96 acres of crop B
- Last row says $P = 19200 - 10X - 8W = 19200$
  Profit is $19,200

4.1: The key was choosing the right pivot

- How did we know 25 was a good pivot?
- In $U = 100 - X - Y$, if we make Y too big, U goes negative
- "too big" is complicated
- If a variable is free, the only safe bet for its value is 0
- But we had $P = 150X + 200Y$, $Y = 0$ is cowardly
- We needed to make Y a pivot, rather than Y being free
- So we want a pivot in the Y column.

4.1: Choosing the pivot

- Choose the pivot column first:
  any column with a negative number at the bottom is OK
  "Leftmost" and "Most negative" are reasonable strategies
- Which row?
- Each row has a maximum allowed Y:
  $U = 100 - X - Y$ allows $Y \leq 100/1 = 100$
  $V = 6000 - 50X - 60Y$ allows $Y \leq 6000/60 = 100$
  $W = 2400 - 20X - 25Y$ allows $Y \leq 2400/25 = 96$
- The W row is most restrictive, so we use it
- After you pick the pivot column, choose the pivot row by computing these ratios

  Choose the smallest non-negative ratio

4.1: Once is not enough

- Usually a single pivot change is not enough
- The bottom row may still have negatives
- Just choose a pivot again, and repeat
- Make sure the right-hand-sides are always non-negative
- If they are negative, problem is harder or you've made a mistake
- See the silly webpage
  `http://www.ms.uky.edu/~jack/2011-08-MA162/ch4.html`