MA111: Contemporary mathematics

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September 7, 2012

Entrance Slip (due 5 min past the hour):

- Can a Borda count winner get no first place votes?
  (Give an example to show it can, or explain why it cannot.)
- Can a Condorcet winner have a majority of last place votes?
  (Give an example to show it can, or explain why it cannot.)

Schedule:

- Online HW 1C,1D,1E,1G is due Friday, Sep 7th, 2012.
- Exam 1 is Monday, Sep 17th, during class.

Today we review our vote counting methods and fairness criteria.
Today we cover IIA and Arrow's theorem.

Review of the vote counting methods

- We have discusssed 4 major (and 2 more minor) vote counting methods:
  - (1.2) **Plurality:** most first place votes wins
  - (1.3) **Borda count:** highest average ranking wins
  - **2nd place is half credit:** like plurality, but 2nd place counts as half a 1st place
  - (1.4) **Plurality with elimination:** eliminate the candidate with the least first place votes
  - **Survivor:** eliminate the candidate with the most last place votes
  - (1.5) **Pairwise comparison:** Most head-to-head victories wins! Ties count half-credit.
- Each method had good features and bad features.
- To be precise, we defined "fairness criteria"
  a vote counting method either satisfied them or not

Review of the fairness criteria

- We have discussed 4 major (and 2 more minor) fairness criterion:
  - (1.2) **Majority (winner) fairness criterion:** If a candidate has more than 50% of the first place votes, he should win.
  - **Majority loser fairness criterion:** If a candidate has more than 50% of the last place votes, he should lose.
  - (1.2) **Condorcet (winner) fairness criterion:** If a candidate can beat every other candidate head-to-head, he should win.
  - **Condorcet loser fairness criterion:** If a candidate is beaten by every other candidate head-to-head, he should lose.
  - (1.4) **Monotonicity:** If a candidate wins one election, then he should also win an election where the only difference is a voter ranked the winner higher. ("more first place votes should help")
  - (1.5) **IIA:** Disqualifying a loser should not affect who wins.

Review: How do they do?

- Here is a table describing how well our vote counting methods do:

	MW	ML	CW	CL	Mo	IIA
PI	Y	N	N	N	Y	N
BC	N	Y	N	Y	Y	N
$2 = \frac{1}{2}$	N	N	N	N	Y	N
PE	Y	*	N	*	N	N
Su	N	Y	N	*	N	N
PC	Y	Y	Y	Y	Y	N

- Notice how none of the methods satisfy all the criteria.
- Kenneth Arrow showed you cannot have a Y in both MW and IIA for any method.
- The * means mathematically no, but practically yes

Activity: Homework help and review

- One side of the worksheet has some leading questions that helped me solve the homework in 1E.
- You should figure out how many total matches are in a pairwise-comparison, and how many total points are awarded.

  Hint: They are the same number!
- The other side is to practice calculating winners.

	5	4	4	3	3	1	1
1st	G	C	B	E	E	D	A
2nd	C	D	D	G	B	B	G
3rd	D	A	E	D	F	A	B
4th	A	F	A	A	A	E	E
5th	B	B	C	F	G	C	C
6th	F	E	F	B	C	F	F
7th	E	G	G	C	D	G	D

- If you finish early, can you tell who is not a winner?
  Harder: Can you invent a method where they DO win? Jack couldn't.

Fast: IIA+Majority in the face of paradox

- In a 3-candidate race where not everyone wins,
  IIA means we can eliminate a loser to get a 2-candidate race
- In a 2-candidate race, there is only one sane way to decide!
- But consider **Condorcet's Paradox:**

	40%	35%	25%
1st	A	B	C
2nd	B	C	A
3rd	C	A	B

- If A is not a winner, then IIA+majority says B wins (75%)
  If B is not a winner, then IIA+majority says C wins (60%)
  If C is not a winner, then IIA+majority says A wins (65%)
- Problem: If B wins, then C is not a winner, so A wins too?...
- Solution: Everyone wins! YAY!

Fast: Conclusion and Arrow's theorem

Theorem (Arrow, 1950)

A vote counting method for more than 2 candidates cannot satisfy both the majority fairness criterion and the IIA fairness criterion.

- A modern proof observes that if the majority rules and only one person thinks A is better than B, then A loses.
- In other words, there is no one that always gets their way.
  "No dictators"
- But phrased pessimistically, this also means in some sense that **each individual is powerless**.
- A clever trick shows that IIA will imply even a majority is powerless.
  But then a majority winner will lose!

Fast: A proof due to Terry Tao, part I

- I expect you to understand a proof of Arrow's theorem. If this one does not work for you, read another online (see last slide for a list).
- First we define the people whose vote doesn't matter:

Definition

Given a set of candidates, a set of voters, and a vote counting method: Call a group of voters **powerless** if whenever that group all agree that A is better than B, but everyone outside the group disagrees, then A loses.

Majority criterion requires a majority to over-rule an individual, so:

Theorem (majority $\implies$ individual is powerless)

If the vote counting method satisfies the majority criterion, then each individual person is powerless.

Fast: Tao's proof part two

The next part has a pretty mathy proof, and hopefully it goes against your intuition a little:

Theorem (IIA $\implies$ powerless+powerless=powerless)

If the vote counting method satisfies IIA, then whenever two powerless groups join together, they are still powerless.

Obviously every group of people is made up of the individuals in it, all of whom are powerless.

Theorem (majority+IIA $\implies$ majority is powerless)

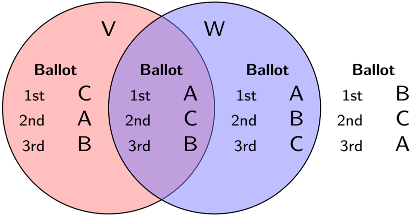
If a vote counting method satisfies both the majority criterion and IIA, then every group of voters is powerless!

Of course, if a majority is powerless, then the majority doesn't rule anymore, so the last theorem really says "It is impossible to satisfy both majority rule and IIA."

Fast: The technical part: powerless+powerless=powerless

Suppose both **V** and **W** are powerless groups of voters that all agree A is better than B, but nobody else does. We need to show A loses.

There is one case where we can do this: Take any third candidate C and imagine the voters have the following very specific ideas about C:



Observe: Everyone in W thinks A is better than C, and everyone else disagrees. Since W is powerless, A loses. Yay! One case done.

However, this actually handles all cases because: Everyone in V thinks C is better than B, and everyone else disagrees. Since V is powerless, C loses. By IIA, C (being a loser) is irrelevant, so no matter how the voters feel about C, A still loses.

Assignment

- Reread and understand pages 2-20, 27-28
- Good book homeworks #1, 3, 17, 23, 33, 59, 60, 61, 62, 68, 72, 73, 74, 75, 79
- Exit slip: Give a single example where each of the following statements is the view of a (sizable) majority:
  - A is better than B
  - B is better than C
  - C is better than D
  - D is better than E
  - E is better than A

  Which candidate is best?

Entrance slip answers

- Can a Borda count winner get no first place votes?

Yes! B wins the Borda count.

	1	1	1
1st	A	D	C
2nd	B	B	B
3rd	C	C	A
4th	D	A	D

- Can a Condorcet winner have a majority of last place votes?

No! In any head-to-head competition, those last place votes will still be a majority of losing votes, so a majority of last place votes means you are a Condorcet **loser**!

Exit slip answers

- Give an example where the best candidates form a circle!

	1	1	1	1	1
1st	A	B	C	D	E
2nd	B	C	D	E	A
3rd	C	D	E	A	B
4th	D	E	A	B	C
5th	E	A	B	C	D

Notice 4/5 prefer A to B, 4/5 prefer B to C, 4/5 prefer C to D, 4/5 prefer D to E, so that we clearly have

$$A \overset{80\%}{>} B \overset{80\%}{>} C \overset{80\%}{>} D \overset{80\%}{>} E$$

so one would think that $A > E$, but no: $E \overset{80\%}{>} A$.

Electing any one candidate woud be a huge mistake.

Alternative proofs of Arrow's theorem

- Yu (2012): Given a preference schedule where A beats B, swap each voter's ranking of A and B in order, until the winner changes. Show that this particular voter decides every election (using a similar trick with C).
- Tao (2009): Opposite proof as here: A "quorum" is the set of people not in a powerless group. Eventually an empty room becomes a quorum and all the voters are powerless.
- Geanakoplos (1996): Three brief proofs. These are similar to Arrow's original proof, but are improved a bit. These versions often appear in more recent textbooks.
- Arrow (1963): Arrow's own book form of the proof. While a short book, it is longer than the other proofs.
- Arrow (1950): Arrow's own original proof. This version made a lot more assumptions, but is also a much more interesting read.