MA111: Contemporary Mathematics
Exam 1: Voting Theory

Name: _____

Date: 2012-09-17

You are not graded on your understanding, but on your communication of your understanding on this exam. Explain your reasoning clearly and concisely. On all but the matching sections, it is necessary to show work (answers alone are worth no credit whatsoever). No network devices of any sort are allowed during the exam. Texting during the exam is considered cheating and will be dealt with according to university policy.

Section 1. Matching for general concepts

____ The options that a group needs to choose from. Often people or actions.

____ The group of people that rank the Candidates, or sometimes the group of criteria used to judge the options.

____ The rankings of the candidates by the voters from best to worst option.

____ The count of how many voters chose each possible ballot.

____ A procedure to choose a candidate based only on the ballots.

____ A way to evaluate a vote counting method by comparing who should win versus who does win.

____ The two fairness criteria Majority and IIA are contradictory.

(a) Candidates

(b) Voters

(c) Ballots

(d) Preference schedule

(e) Vote Counting Method

(f) Fairness Criterion

(g) Arrow's theorem

Section 2. Matching for vote counting methods

____ Each first place vote gets 1 point. Most points wins.

____ Each first place vote gets 2 points, and each second place vote gets 1 point. Most points wins.

____ Each last place vote gets 1 point, each second to last place vote gets 2 points, etc. Most points wins.

____ Repeatedly eliminate the candidate with the most last place votes, until only one is left.

____ Repeatedly eliminate the candidate with the least first place votes, until only one is left.

____ For each head-to-head matchup: award the victor 1 point or divide the oint evenly if they tie. Most points wins.

(a) Plurality

(b) Second-place is half-credit

(c) Borda Count

(d) Survivor

(e) Plurality with elimination

(f) Pairwise-comparison

Section 3. Matching for fairness criteria

_____ A candidate with more than half of the first place votes should win.

_____ A candidate with more than half of the last place votes should lose.

_____ A candidate that can be every other candidate head-to-head should win.

_____ A candidate that loses to every other candidate head-to-head should lose.

_____ If a voter changes their ranking to move the winning candidate up, but leaves the others unchanged in their relative order, then the candidate who is moved up should still win.

_____ If a candidate loses then people can rank that candidate however they want, but the winner should either not change, or should include the (old) loser.

(a) Majority

(b) Majority loser

(c) Condorcet

(d) Condorcet loser

(e) Monotonicity

(f) IIA

Section 4. Vote counting methods

Use the following preference schedule to answer the questions. You must show your work. Clearly communicate that you understand the vote counting method or you will not receive credit.

	3	2	2	1
1st	A	C	D	B
2nd	B	D	B	D
3rd	C	A	C	C
4th	D	B	A	A

1. Who wins according to the plurality vote counting method?

2. Who wins according to the Borda count vote counting method?

3. Who wins according to the plurality with elimination vote counting method?

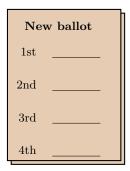4. Who wins according to the pairwise-comparison vote counting method?

Section 5. Strategic Voting

In each of the following elections, the left-most column of voters is unhappy with the result. If they had a chance to change their own votes, how could they get a better result? Be sure to explain who wins the new election and why the left-most column of voters is happier with the result.
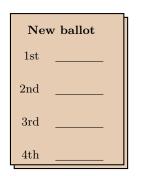
	2	10	11
1st	A	B	D
2nd	B	A	A
3rd	C	D	C
4th	D	C	B

1. The election is decided by plurality.

New ballot

1st _____

2nd _____

3rd _____

4th _____

	2	10	99
1st	A	B	D
2nd	B	A	A
3rd	C	D	C
4th	D	C	B

2. The election is decided by plurality.

New ballot

1st _____

2nd _____

3rd _____

4th _____

	3	2	2
1st	A	C	D
2nd	B	B	B
3rd	C	A	A
4th	D	D	C

3. Election decided by Borda Count.

New ballot

1st _____

2nd _____

3rd _____

4th _____

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4. Election decided by Plurality with Elimination.

	1	3	2	2	1
1st	A	A	B	C	D
2nd	B	B	C	A	C
3rd	C	C	A	B	A
4th	D	D	D	D	B

New ballot

1st _____

2nd _____

3rd _____

4th _____

5. Election decided by Pairwise Comparison.

	2	3	2	1
1st	B	A	C	B
2nd	C	B	D	D
3rd	A	C	A	A
4th	D	D	B	C

New ballot

1st _____

2nd _____

3rd _____

4th _____

Section 6. Fairness criteria

For each combination, either give an example where it does happen, or give a clear argument for why it can never happen. You can restrict to 3 or 4 candidates if this is helpful.

1. Can a majority winner be a majority loser?

2. Can a majority winner lose a plurality election?

3. Can a majority winner lose a borda count election?

4. Can a majority winner lose a plurality with elimination election?

5. Can a majority winner be a condorcet winner?

6. Can a majority winner have the most last place votes?

7. Can a condorcet winner be a majority winner?

8. Can a Condorcet winner have no first place votes?

9. Can a Condorcet loser have the most first place votes?

Answer Key for Exam A

Section 1. Matching for general concepts

(a) The options that a group needs to choose from. Often people or actions.

(b) The group of people that rank the Candidates, or sometimes the group of criteria used to judge the options.

(c) The rankings of the candidates by the voters from best to worst option.

(d) The count of how many voters chose each possible ballot.

(e) A procedure to choose a candidate based only on the ballots.

(f) A way to evaluate a vote counting method by comparing who should win versus who does win.

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(c) Each last place vote gets 1 point, each second to last place vote gets 2 points, etc. Most points wins.

(d) Repeatedly eliminate the candidate with the most last place votes, until only one is left.

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(f) For each head-to-head matchup: award the victor 1 point or divide the oint evenly if they tie. Most points wins.

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(e) If a voter changes their ranking to move the winning candidate up, but leaves the others unchanged in their relative order, then the candidate who is moved up should still win.

(f) If a candidate loses then people can rank that candidate however they want, but the winner should either not change, or should include the (old) loser.

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(f) IIA

Section 4. Vote counting methods

Use the following preference schedule to answer the questions. You must show your work. Clearly communicate that you understand the vote counting method or you will not receive credit.

	3	2	2	1
1st	A	C	D	B
2nd	B	D	B	D
3rd	C	A	C	C
4th	D	B	A	A

1. Who wins according to the plurality vote counting method?

   Answer: A with 3 first place votes, the most amongst the others (B:1, C:2, D:2)

2. Who wins according to the Borda count vote counting method?

   Answer: B with $22 = (3)(3) + (2)(1) + (2)(3) + (1)(4)$ points, the most amongst the others (A:19, C:20, D:20)

3. Who wins according to the plurality with elimination vote counting method?

   Answer: D with B eliminated first because it has 1 first place vote, the least amongst the others (A:3, C:2, D:2), then C eliminated second because it has 2 "first" place votes, the least amongst the others (A:3, D:$2 + 1$), then A eliminated third because it has 3 "first" place votes, the least amongst the others (D:$2 + 2 + 1$)

   Notice how eliminating a candidate changes who ends up getting the "first" place votes, since those votes go to the highest ranked candidate amongst those still in the running.

4. Who wins according to the pairwise-comparison vote counting method?

   Answer: C with 2 points form 2 wins, the highest amongst the others (A:1 (1W), B:1.5 (1W,1T), D:1.5(1W,1T)). The results were: A̲vB, Av C̲, Av D̲, B̲vC, B̲vD̲, C̲vD

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Section 5. Strategic Voting

In each of the following elections, the left-most column of voters is unhappy with the result. If they had a chance to change their own votes, how could they get a better result? Be sure to explain who wins the new election and why the left-most column of voters is happier with the result.

1. The election is decided by plurality.

	2	10	11
1st	A	B	D
2nd	B	A	A
3rd	C	D	C
4th	D	C	B

Answer:

New ballot	
1st	B
2nd	A
3rd	C
4th	D

The left column of two voters are unhappy with the current winner D (A:2, B:10, C:0, D:11), their least favorite. If they change their ballot as indicated, then B will win (A:0, B:12, C:0, D:11), which is much better for the first column than D.

Warning: the left column is not allowed to change other people's votes! Even though the average (or total) happiness would likely be higher if everyone voted for A, we are only asking about how the left two voters can change their votes.

2. The election is decided by plurality.

	2	10	99
1st	A	B	D
2nd	B	A	A
3rd	C	D	C
4th	D	C	B

Answer:

New ballot	
1st	☹
2nd	☹
3rd	☹
4th	☹

They can't! The left column of two voters are unhappy with the current winner D (A:2, B:10, C:0, D:99), their least favorite. If they change their ballot then nothing changes, because D has a huge majority.

Warning: the left column is not allowed to change other people's votes! Even though the average (or total) happiness would likely be higher if everyone voted for A, we are only asking about how the left two voters can change their votes.

3. Election decided by Borda Count.

	3	2	2
1st	A	C	D
2nd	B	B	B
3rd	C	A	A
4th	D	D	C

Answer:

New ballot	
1st	A
2nd	C
3rd	B
4th	D

B wins the Borda count (A:20, B:21, C:16, D:13) which shouldn't bother the first column of voters too much, but maybe they get greedy and decide A must win! In the new election A wins (A:20, B:18, C:19, D:13) which the first column prefers.

4. Election decided by Plurality with Elimination.

	1	3	2	2	1
1st	A	A	B	C	D
2nd	B	B	C	A	C
3rd	C	C	A	B	A
4th	D	D	D	D	B

C wins the plurality with elimination (D eliminated, then B, then A), but the voter from the first column would prefer A or B to win. If he goes completely crazy, he could vote in the exact opposite order as he truly believes. This gives D an extra vote, and suddenly B,C,D all tie for first place votes, and so all three are eliminated! This leaves A the winner, the crazy voter's true favorite. This is an example of a violation of monotonicty. Why did this

work? By the end of the method, C is much too strong for A. A's only chance is for the other candidates to take out C early. Since D was so close to remaining in the race, a vote for D is a vote for "keep after C". This new election ran like the "cornucopia" from the hunger games.

Answer:

New ballot	
1st	D
2nd	C
3rd	B
4th	A

5. Election decided by Pairwise Comparison.

	2	3	2	1
1st	B	A	C	B
2nd	C	B	D	D
3rd	A	C	A	A
4th	D	D	B	C

Answer:

New ballot	
1st	B
2nd	C
3rd	D
4th	A

The original election is pretty close. A:2.5 (2W,1T), B: 2 (2W), C: 1.5 (1W,1T) D:0. A wins. Since pairwise comparison is monotonic we know that we cannot hurt A by raising it (and in reverse, we cannot make B win by lowering B). Hence we need to appeal to IIA and help a loser so that A loses. We do this by sending A to the bottom. Now the AvD goes to D instead of A, and the scores are A:1.5 (1W,1T), B:2 (2W), C:1.5 (1W,1T), D:1 (1W), and B wins!

Section 6. Fairness criteria

For each combination, either give an example where it does happen, or give a clear argument for why it can never happen. You can restrict to 3 or 4 candidates if this is helpful.

1. Can a majority winner be a majority loser?

   Answer: No. If more than half of the voters rank a candidate as best (so that the candidate is a majority winner), then less than half of the voters remain to rank the candidate last place (so that the candidate cannot be a majority loser).

2. Can a majority winner lose a plurality election?

   Answer: No. If more than half of the voters rank a candidate as best (so that the candidate is a majority winner), then less than half of the voters remain to rank the other candidates first place (so they will all have fewer first place votes than the majority winner). Since the plurality winner is the candidate with the most first place votes, the majority winner will win.

3. Can a majority winner lose a borda count election?

   Answer: Yes. If there are many candidates and the majority winner also has many last place votes, then a moderate candidate with many second place votes can still win. For example in the right hand election, A gets $(60)(4) + (40)(1) = 280$ points, while B gets $(60)(3) + (40)(4) = 340$ points, so A must lose:

	60	40
1st	A	B
2nd	B	C
3rd	C	D
4th	D	A

4. Can a majority winner lose a plurality with elimination election?

   Answer: No. If more than half of the voters rank a candidate first, then that candidate is never eligible for elimination. Since someone has to be eliminated, eventually the only candidate left is the majority winner.

5. Can a majority winner be a condorcet winner?

   Answer: Yes. In fact, a majority winner is always a condorcet winner! If more than half of the voters rank a candidate as best (so a majority winner), then in any head-to-head match, those same voters also ensure that same candidate wins (so the majority winner wins every head-to-head match, making him a Condorcet winner). Since any example will work, we give a simple unanimous example:

	100
1st	A
2nd	B
3rd	C
4th	D

   A wins every head-to-head 100-to-0.

6. Can a majority winner have the most last place votes?

   Answer: Yes. If there are enough candidates to spread the burden of the last place votes, the majority winner can have the most last place votes. For example here the majority winner (A with 60%) has twice as many last place votes (40%) as the others have (20% each):

	20	20	20	40
1st	A	A	A	B
2nd	B	C	D	C
3rd	C	D	B	D
4th	D	B	C	A

7. Can a condorcet winner be a majority winner?

   Answer: Yes. For example it could be unaninmous that A is best:

	70	20	10
1st	A	A	A
2nd	B	C	D
3rd	C	B	B
4th	D	D	C

8. Can a Condorcet winner have no first place votes?

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Answer: Yes. If first place votes are split between several candidates, and another candidate has all second place votes, then no single candidate will do better in a head-to-head. For example in the following election, Z versus anyone is 75-to-25.

	25	25	25	25
1st	A	B	C	D
2nd	Z	Z	Z	Z
3rd	B	C	D	A
4th	C	D	A	B
5th	D	A	B	C

9. Can a Condorcet loser have the most first place votes?

Answer: Yes. If first place votes are split amongst many candidates, then a small number is enough to be "most". The rest can be last-place votes. For example in the following election, A loses each head-to-head 34-to-66, but it has the most first place votes.

	34	33	33
1st	A	B	C
2nd	B	C	B
3rd	C	A	A