MA162: Finite mathematics

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Schedule:

- HW 4.1 is due Friday, Oct 12th, 2012
- Exam 2 is Monday, Oct 15th, 5:00pm-7:00pm in BS107 and BS116.

Today we will cover 4.1: solving real problems with RREF

4.1: Linear programming problems

- An LPP has three parts:
  - The variables (the business decision to be made)
  - The inequalities (the laws, constraints, rules, and regulations)
  - The objective (maximize profit, minimize cost)
- If there are more than two variables, graphing gets hard!
- We need to use matrices to handle so many variables.

4.1: Today's LPP as word problem

- Old MacDonald had 100 acres, \$6000, and 2400 labor hours
- Crop A costs him \$50/acre and 20hrs/acre in labor
- Crop B costs him \$60/acre and 25hrs/acre in labor
- Crop A earns him \$150/acre and Crop B earns him \$200/acre
- How many acres of each crop should he plant?

4.1: Today's LPP in summary form

- Variables:

  X = Number of acres of crop A to plant
  Y = Number of acres of crop B to plant

- Constraints:

$$\begin{aligned} X + Y &\leq 100 && \text{Land} \\ 50X + 60Y &\leq 6000 && \text{Capital} \\ 20X + 25Y &\leq 2400 && \text{Labor} \end{aligned}$$

- Objective:

  Maximize $P = 150X + 200Y$

4.1: Inequalities are just equalities in disguise

- All of our variables are non-negative (realistic)
- Think about $X + Y \leq 100$
- It means $X + Y$ has not yet exceeded 100
- We could still add something to bump it up to 100
- $X + Y + U = 100$ with $X, Y, U \geq 0$
- U is "the rest" of the 100, the "unused" land, the **slack**

$$U = 100 - (X + Y)$$

- "$U = 20$" means we left 20 acres fallow (unused)

4.1: LPP are just systems of equations in disguise

$$\begin{aligned} X + Y &\leq 100 && \text{Land} \\ 50X + 60Y &\leq 6000 && \text{Capital} \\ 20X + 25Y &\leq 2400 && \text{Labor} \end{aligned}$$

- Define our slack variables:

$$\begin{aligned} U &= 100 - (X + Y) && \text{unused Land} \\ V &= 6000 - (50X + 60Y) && \text{unused Capital} \\ W &= 2400 - (20X + 25Y) && \text{unused Labor} \end{aligned}$$

- Even profit is an equation: $P = 150X + 200Y$ just means $-150X - 200Y + P = 0$
- Now we have a system of equations:

$$\begin{aligned} X + Y + U &= 100 && \text{Land} \\ 50X + 60Y + V &= 6000 && \text{Capital} \\ 20X + 25Y + W &= 2400 && \text{Labor} \\ \hline -150X - 200Y + P &= 0 && \text{Profit} \end{aligned}$$

4.1: Write it as a matrix

- The system of equations:

$$\begin{array}{rrrrrrl}
X & + & Y & + & U & = 100 & \text{Land} \\
50X & + & 60Y & + & V & = 6000 & \text{Capital} \\
20X & + & 25Y & + & W & = 2400 & \text{Labor} \\
\hline
-150X & - & 200Y & + & P & = 0 & \text{Profit}
\end{array}$$

- Now as a matrix:

$$\begin{array}{c}
\begin{array}{cccccc}
X & Y & U & V & W & P
\end{array} \\
\left(\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 0 & 0 & 100 \\
50 & 60 & 0 & 1 & 0 & 0 & 6000 \\
20 & 25 & 0 & 0 & 1 & 0 & 2400 \\
\hline
-150 & -200 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
\end{array}
\begin{array}{l}
\\
\text{Land} \\
\text{Capital} \\
\text{Labor} \\
\\
\text{Profit}
\end{array}$$

4.1: Analyze the matrix

- Our matrix is basically in RREF!

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P & RHS \end{array} \\ \left(\begin{array}{cccccc|c} 1 & 1 & \textcircled{1} & 0 & 0 & 0 & 100 \\ 50 & 60 & 0 & \textcircled{1} & 0 & 0 & 6000 \\ 20 & 25 & 0 & 0 & \textcircled{1} & 0 & 2400 \\ \hline -150 & -200 & 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Labor} \\ \text{Profit} \end{array}$$

- U,V,W,P have pivots, and X,Y are free.
  We can make X and Y whatever we want!

- Wait. What do we want them to be?
  Isn't that why we started doing this anyways?

- OMG We have the wrong pivots!

4.1: Operation Row Op to the rescue!

- Well, what if we decided the 25 should have been a pivot?
- Easy to fix using row ops:

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ 50 & 60 & 0 & 1 & 0 & 0 & 6000 \\ 20 & 25 & 0 & 0 & 1 & 0 & 2400 \\ \hline -150 & -200 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_4+8R_3 \\ R_3/25}]{\substack{R_1-0.04R_3 \\ R_2-2.4R_3}} \left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array}\right)$$

- Y, U, V, P are pivots, X and W are free.

4.1: What does it say now?

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P \end{array} \\ \left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Crop B} \\ \text{Profit} \end{array}$$

- First row says $U = 4 - (1/5)X + (1/25)W$
- Last row says $P = 19200 - 10X - 8W$
- X and W are free, what should they be?
- Every acre of crop A we plant costs us \$10!

4.1: Summary

$$\begin{array}{c} \begin{array}{cccccc} X & Y & U & V & W & P \end{array} \\ \left(\begin{array}{cccccc|c} 1/5 & 0 & \textcircled{1} & 0 & -1/25 & 0 & 4 \\ 2 & 0 & 0 & \textcircled{1} & -12/5 & 0 & 240 \\ 4/5 & \textcircled{1} & 0 & 0 & 1/25 & 0 & 96 \\ \hline 10 & 0 & 0 & 0 & 8 & \textcircled{1} & 19200 \end{array}\right) \end{array} \begin{array}{l} \\ \text{Land} \\ \text{Capital} \\ \text{Crop B} \\ \text{Profit} \end{array}$$

- Our free variables are $X = 0$ and $W = 0$
  Plant no acres of crop A, and use all available labor
- First row says $U = 4 - (1/5)X + (1/25)W = 4$
  Leave 4 acres fallow
- Second row says $V = 240 - 2X + (12/5)W = 240$
  Leave $240 unspent
- Third row says $Y = 96 - (4/5)X - (1/25)W = 96$
  Plant 96 acres of crop B
- Last row says $P = 19200 - 10X - 8W = 19200$
  Profit is $19,200

4.1: The key was choosing the right pivot

- How did we know 25 was a good pivot?
- In $U = 100 - X - Y$, if we make Y too big, U goes negative
- "too big" is complicated
- If a variable is free, the only safe bet for its value is 0
- But we had $P = 150X + 200Y$, $Y = 0$ is cowardly
- We needed to make Y a pivot, rather than Y being free
- So we want a pivot in the Y column.

4.1: Choosing the pivot

- Choose the pivot column first:
  any column with a negative number at the bottom is OK
  "Leftmost" and "Most negative" are reasonable strategies
- Which row?
- Each row has a maximum allowed Y:
  $U = 100 - X - Y$ allows $Y \leq 100/1 = 100$
  $V = 6000 - 50X - 60Y$ allows $Y \leq 6000/60 = 100$
  $W = 2400 - 20X - 25Y$ allows $Y \leq 2400/25 = 96$
- The W row is most restrictive, so we use it
- After you pick the pivot column, choose the pivot row by computing these ratios

  Choose the smallest non-negative ratio

4.1: Once is not enough

- Usually a single pivot change is not enough
- The bottom row may still have negatives
- Just choose a pivot again, and repeat
- Make sure the right-hand-sides are always non-negative
- If they are negative, problem is harder or you've made a mistake
- See the silly webpage
  `http://www.ms.uky.edu/~jack/2011-08-MA162/ch4.html`