MA322-001 Jan 22 worksheet

Vector geometry!

One trick to getting used to linear algebra is to combine geometric intuition with the number crunching (we've done) and algebraic manipulations (to come). This worksheet is meant to be read from top to bottom, solve part (d), and then back up again. It is also example 4 in the book.

(a) Can you solve this system of equations? I guess.

$$\begin{cases} -1x_1 + 2x_2 = 8 \\ 1x_1 + 1x_2 = 1 \end{cases}$$

new $R_2 = R_2 + R_1$ looks good

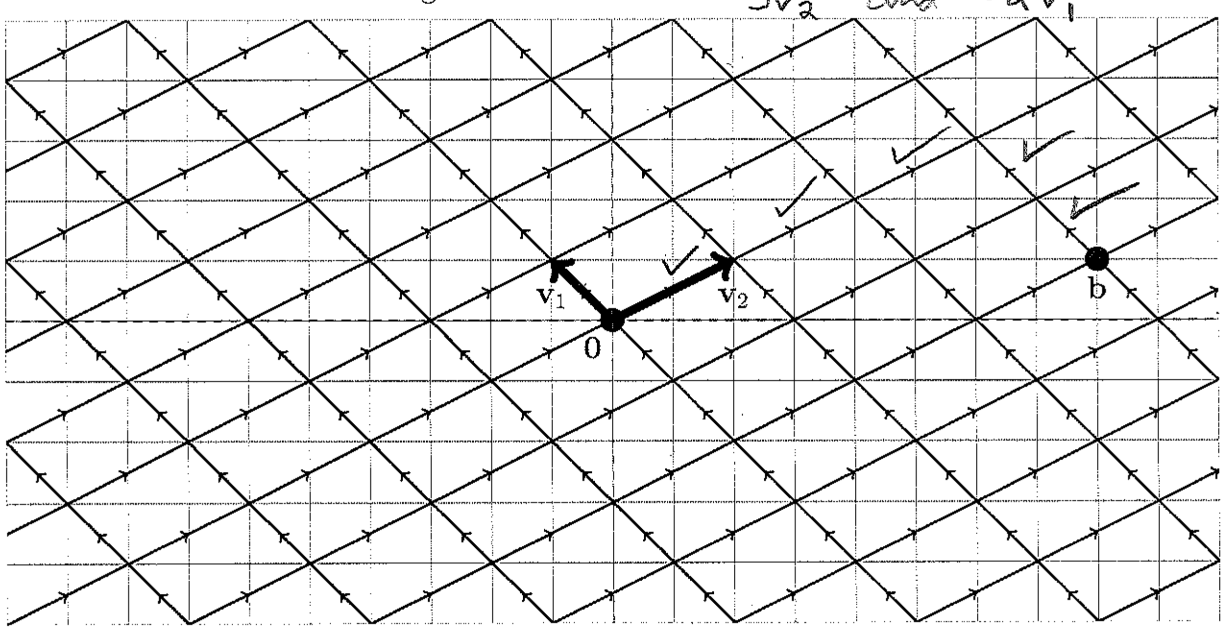
(b) Can you solve this vector equation? Same as (a)

$$x_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

(c) Can you write b as a linear combination $x_1 v_1 + x_2 v_2 = b$ where $b = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$? Literally the same as b! Just named stuff.

(d) Can you write b as a linear combination $x_1 v_1 + x_2 v_2 = b$ using this picture and no calculation other than counting?

$3v_2$ and $-2v_1$



Stoichiometry!

This worksheet is meant to be read from top to bottom and the back to the top. The answers are likely to be "no" or "I don't know" on the way down, but should be better on the way back up.

(a) Can you balance this reaction? No?

$$\underbrace{\qquad}CH_4 + \underbrace{\qquad}CO_2 \to \underbrace{\qquad}H_2O$$

(b) Can you solve this vector equation? Maybe?

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(c) Can you solve this system of equations? Probably. $\{x_1 + x_2 = 0,\ 4x_1 = 2x_3,\ 2x_2 = x_3\}$

(d) Can you row reduce this matrix? Definitely!

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & \# \\ \hline 1 & 1 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

new $R_2 = R_2 - 4R_1$
new $R_2 = -\frac{1}{4} R_2$
new $R_3 = R_3 - 2R_2$
new $R_3 = -\frac{1}{2} R_3$
new $R_2 = R_2 - \frac{1}{2} R_3$
new $R_1 = R_1 - R_2$
RREF!

See attached page for work
Part (e) has the final matrix

(e) What is the general solution to the system of equations with this augmented matrix? Easy!

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & \# \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Equations $\left\{ \begin{array}{l} X_1 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{array} \right\}$, and solutions $\left\{ \begin{array}{l} X_1 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{array} \right\}$

(f) Now go back and answer (c), (b), and (a).
(c) Hey, $x_1 = x_2 = x_3 = 0$ works! (b) $X_1 = X_2 = X_3 = 0$
Weird only 0 works. Most chemists would be upset.
Apparently my methane + carbondioxide rain maker isn't ready yet.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{new } R_2 = R_2 - 4R_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{new } R_2 = -\frac{1}{4} R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{new } R_3 = R_3 - 2R_2}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{\text{new } R_3 = -\frac{1}{2} R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{new } R_2 = R_2 - \frac{1}{2} R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{new } R_1 = R_1 - R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Harder Stoichiometry. This problem was from a chemistry class with a MA322 pre-requisite.

Read me: Convince yourself that solving (a) is the same as solving (b), which is the same as solving (c), and that (d) would be all the actual calculation, so that the answer to (e) is very useful, but (f) is good enough. Now answer (e), (f) and (a).

(a) Balance this reaction

$$\underbrace{\quad}\,P_2I_4 + \underbrace{\quad}\,P_4 + \underbrace{\quad}\,H_2O = \underbrace{\quad}\,PH_4I + \underbrace{\quad}\,H_3PO_4$$

(b) Solve this vector equation:

$$x_1 \begin{bmatrix} 0 \\ 4 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = x_4 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \quad \begin{matrix} H \\ I \\ O \\ P \end{matrix}$$

(c) Solve this system of equations:

$$\begin{cases} H & 2x_3 = 4x_4 + 3x_5 \\ I & 4x_1 = x_4 \\ O & x_3 = 4x_5 \\ P & 2x_1 + 4x_2 = x_4 + x_5 \end{cases}$$

(d) Show that the matrix A is row equivalent to the matrix B (personally, I'd ask a computer to do this; you could just believe me that it is true during class, and check it at home in a spreadsheet).

$$A = \begin{bmatrix} 0 & 0 & 2 & -4 & -3 & 0 \\ 4 & 0 & 0 & -1 & -0 & 0 \\ 0 & 0 & 1 & -0 & -4 & 0 \\ 2 & 4 & 0 & -1 & -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 & -10/32 & 0 \\ 0 & 1 & 0 & 0 & -13/32 & 0 \\ 0 & 0 & 1 & 0 & -128/32 & 0 \\ 0 & 0 & 0 & 1 & -40/32 & 0 \end{bmatrix}$$

(e) Describe all possible solutions to the system of equations whose augmented matrix is B.

$X_1 = 10/32\ x_5$ $\qquad$ x_5 is free

$X_2 = 13/32\ x_5$

$X_3 = 128/32\ x_5$

$x_4 = 40/32\ x_5$

(f) What is a solution in which all the variables are integers?

Set $x_5 = 32$, then

$X_1 = 10$, $x_2 = 13$, $X_3 = 128$, $x_4 = 40$, $x_5 = 32$

works for (c), (b), and (a)

MA322-001 Jan 22 quiz

Name: Key

1.2.1 (HW1.2#13) Describe the general solution to the system of equations represented by the following augmented matrix. Make sure your solution has no circular or nested definitions (free variables should be labelled as free, basic variables should be defined in terms of free variables and not in terms of other basic variables; look these words up in 1.2 if you don't know what they mean).

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \# \\ \hline 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 3x_5 - 2 \\ x_2 = 4x_5 + 1 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$$

new $R_1 = R_1 + R_3 + 3R_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 3x_5 = -2$$
$$x_2 - 4x_5 = 1$$
$$x_4 + 9x_5 = 4$$

1.3.1 Write $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as a linear combination of $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Easy to use picture

2 v_2's and $2v_1$'s get you there.

$$2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \checkmark$$

1.3.2 Balance this equation: $\underbrace{1}\, C_3H_8 + \underbrace{3}\, O_2 \to \underbrace{3}\, CO_2 + \underbrace{4}\, H_2O$

$$\begin{array}{c} C \\ H \\ O \end{array}\left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{array}\right] \xrightarrow{\text{new } R_1 = 3R_1 - R_2 \text{ (cheaty)}} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{array}\right] \xrightarrow{R_2 - 8R_1} \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 24 & -18 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -3/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{bmatrix} \qquad x_1 = \frac{1}{4}x_4, \quad x_2 = \frac{3}{4}x_4, \quad x_3 = \frac{3}{4}x_4$$

You should get this back on Fri Jan 24. The following note is for you to read then:

Before next class (Mon Jan 27) (a) reread 1.4 and fix your notes, (b) read 1.5 and get your notes ready for next class, and (c) do HW1.4 #1,3,5*,7*,13,25 For 5 and 7 answer the question, but notice there is no calculation required. You may find that 5, 7, and 25 go together nicely.