2.3: Matrix invertibility

MA322-001 Feb 14 Worksheet

How are pivots related to linearly independent columns?

The set of pivot columns is lin. ind. Any free columns mean they are dependent on the pivot columns.

How are linearly dependent columns related to $A\vec{x} = 0$?

Literally the same! $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = 0$ iff $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \cdots \vec{v}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$ | Dep iff Nonzero solution iff free column

How is $A\vec{x} = 0$ (with $\vec{x}$ and 0 vectors) related to $AX = 0$ (with X and 0 matrices)?

If $A\vec{x} = 0$ and $A\vec{y} = 0$ then $A[\vec{x}|\vec{y}] = [0|0]$

How is $AX = 0$ related to $BA = I$ with I the identity matrix?

Can't have both! $B(AX) = B(0) = 0$, $(BA)X = IX = X$ so $X = 0$ is only solution

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How are pivots related to linearly independent rows?

Pivot Rows in Echelon Form are ind.
Zero Rows in Echelon Form came from row dependency

How are linear depdendent rows related to $A\vec{x} = \vec{b}$?

If the rows of A are dependent, then the rows of $\vec{b}$ satisfy the same
if $R_1 + R_2 = R_3$ then $A\vec{x} = \vec{b}$ requires $b_1 + b_2 = b_3$

How is $A\vec{x} = \vec{b}$ related to $AB = I$ with I the identity matrix?

If $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2$, ... then $B = [\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n]$

How is $AB = I$ related to the span of the columns of A?

It requires the span to be all of $\mathbb{R}^n$ since it contains $\vec{e}_1 \cdots \vec{e}_n$

For a square matrix A how is $A\vec{x} = \vec{b}$ related to having linearly indepdent columns whose span is all of $\mathbb{R}^n$?

Unique solution for each $\vec{b}$ (1-1, onto)

How is $AX = I$ related to that?

Same, if A is square. Onto

How is $BA = I$ related to that?

Same, if A is square. 1-1

How are B and X related if A is square?

$B = X$

MA322-001 Feb 14 Quiz Name: Key

2.2.1(HW2.2#1) Find a matrix X such that $AX = I$ where $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 8 & 6 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -\frac{5}{2} & 4 \end{array}\right] \quad \text{so} \quad X = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$$

2.2.2 (a) Find a vector $\vec{c}$ such that $I\vec{c} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \vec{c} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ because $I\vec{c} = \vec{c}$

(a) What is $X\vec{c}$? $\quad \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4+3 \\ -5-4 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$

2.2.3 (HW2.2#5) Find the solution to $\begin{cases} 8x_1 + 6x_2 = \ \ 2 \\ 5x_1 + 4x_2 = -1 \end{cases}$ $\quad x_1 = 7$, $x_2 = -9$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \vec{c} = I\vec{c} = A(X\vec{c}) = A\begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

2.2.3 (HW2.2#33) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_5 - R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3}$

Now Do these same ops to Identity

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \xrightarrow{R_5 - R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

2.3.1 Find a matrix with 2 rows and 4 columns with linearly independent rows, and linearly dependent columns.

$$\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} \quad \text{for any } a, b, c, d$$