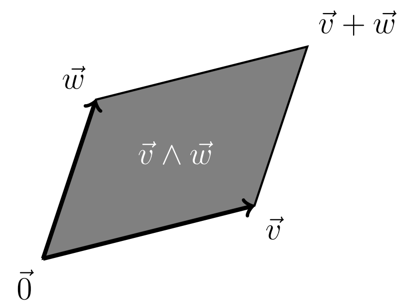
3.3: Matrix determinants as volume MA322-001 Feb 17 Worksheet

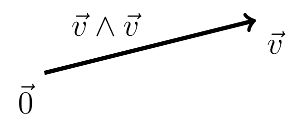
Today is only square matrices. Pictures and intuition will be for 2×2.

There is a very important construction in linear algebra that goes by many names: "wedge," "exterior product," "Grassmann product", and "alternating forms" amongst others. It is written $\vec{v} \wedge \vec{w}$. In the 2×2 case it means the area* of the parallelogram with corners at $\vec{0}$, $\vec{v}$, $\vec{w}$, and $\vec{v}+\vec{w}$.

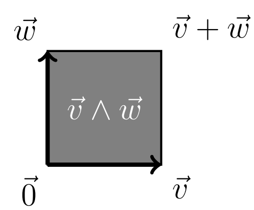


Explain each of the following axioms:

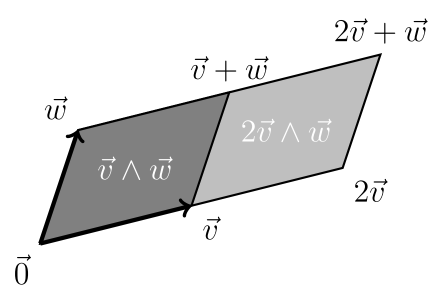
Axiom 0: $\vec{v} \wedge \vec{v} = 0$



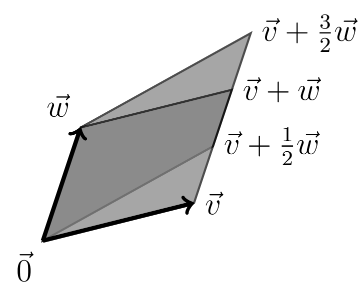
Axiom 1: $\vec{e}_1 \wedge \vec{e}_2 = 1$ where $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$



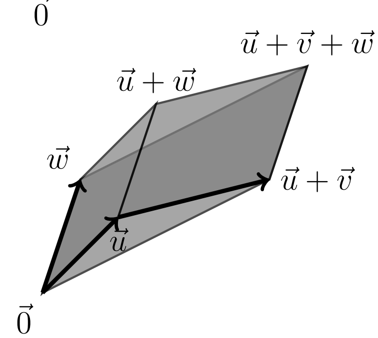
Axiom 2: $(c\vec{v}) \wedge \vec{w} = c(\vec{v} \wedge \vec{w})$ and
$\vec{v} \wedge (c\vec{w}) = c(\vec{v} \wedge \vec{w})$



Axiom 3': $\vec{v} \wedge (\vec{w} + c\vec{v}) = \vec{v} \wedge \vec{w}$ and
$(\vec{v} + c\vec{w}) \wedge \vec{w} = \vec{v} \wedge \vec{w}$



Axiom 3: $(\vec{u}+\vec{v}) \wedge \vec{w} = \vec{u} \wedge \vec{w} + \vec{v} \wedge \vec{w}$ and
$\vec{u} \wedge (\vec{v}+\vec{w}) = \vec{u} \wedge \vec{v} + \vec{u} \wedge \vec{w}$



Chapter 3.2: Calculating determinants

The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the area $\begin{bmatrix} a \\ c \end{bmatrix} \wedge \begin{bmatrix} b \\ d \end{bmatrix}$ and the area of $(a, b) \wedge (c, d)$.

[Gift certificate challenge: why are those areas the same?]

The determinant turns out to be $ad - bc$ (in both cases, rows and columns).

The determinant of an $n \times n$ matrix does not have a simple formula that can be computed efficiently (it has a formula that takes about $n!$ operations to compute). However it is defined similarly:

Axiom 0: $\ldots \wedge \vec{v} \wedge \vec{v} \wedge \ldots = 0$ (Alternating in each position; Collapse)

Axiom 1: $\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n = 1$ (Unit hypercube)

Axiom 2: $\ldots \wedge (c\vec{v}) \wedge \ldots = c(\ldots \wedge \vec{v} \wedge \ldots)$ (Scaling in each position)

Axiom 3: $\ldots \wedge (\vec{u} + \vec{v}) \wedge \ldots = (\ldots \wedge \vec{u} \wedge \ldots) + (\ldots \wedge \vec{v} \wedge \ldots)$ (Additive in each position)

The determinant $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \ldots \\ \vec{v}_n \end{bmatrix}$ is defined to be $\vec{v}_1 \wedge \vec{v}_2 \wedge \ldots \wedge \vec{v}_n$, the volume of the "parallelepiped" spanned by the rows.

How do we compute it? The axioms tell us how elementary row ops affect the volume: Axiom 3 says that "add a multiple of one row to another" does not change the volume; Axiom 2 says scaling a row scales the volume; Axiom 1 says that once we get the matrix to RREF we know the volume; Axiom 0 and 3 work together to say that a row-swap negates the signed volume.

For instance what is the volume of the parallelepiped spanned by the rows of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 12 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 12 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 - R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

This has volume (1)(2)(3) since the row ops did not change the volume, and the last matrix is a rectangular prism (box) with length 1, width 2, and height 3.

What is the volume of the parallelepiped spanned by the rows of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 20 \\ 5 & 10 & 20 \end{bmatrix}$?

MA322-001 Feb 17 Quiz Name:________________________

2.3.1 (HW2.3#3/13) Is the matrix $\begin{bmatrix} -3 & 5 & 8 \\ 0 & -4 & -3 \\ 0 & 0 & 3 \end{bmatrix}$ invertible?

2.3.2 (HW2.3#5) Is the matrix $\begin{bmatrix} 3 & 0 & -3 \\ 2 & 0 & 4 \\ -4 & 0 & 7 \end{bmatrix}$ invertible?

3.3.1 What is the volume of the parallelepiped spanned by the rows of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 20 \\ 5 & 10 & 20 \end{bmatrix}$?

3.3.2 What is the area of the triangle with vertices $(1, 2)$, $(5, 3)$, and $(6, 8)$?