4.2a: Null spaces

Define Nul(A) to be the solution set $\{\vec{x} : A\vec{x} = \vec{0}\}$ to the homogeneous equation $A\vec{x} = 0$. It turns out that Nul(A) is always a subspace, no matter which matrix A is.

Verify the three part test to be a subspace here:

(a) Is $\vec{0}$ in Nul(A)? In other words, is $A\vec{0} = \vec{0}$?
Yes! $A\vec{0} = \vec{0}$ no matter what matrix A is.

(b) If $\vec{v}$ in Nul(A), is $c \cdot \vec{v}$ in Nul(A)? In other words, if $A(\vec{v}) = \vec{0}$, does $A(c\vec{v}) = \vec{0}$? Yes! $A(c\vec{v}) = c(A\vec{v}) = c\vec{0} = \vec{0}$

(c) If $\vec{v}, \vec{w}$ in Nul(A), is $\vec{v} + \vec{w}$ in Nul(A)? In other words, if $A\vec{v} = A\vec{w} = \vec{0}$, does $A(\vec{v} + \vec{w}) = \vec{0}$? Yes! $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \vec{0} + \vec{0} = \vec{0}$

It is not to hard to test if a vector $\vec{v}$ is in Nul(A): just multiply it by A and check that you get $\vec{0}$. On the other hand, how do you find lots of vectors in Nul(A)? RREF!

Explain how to get all vectors in Nul(A) for a matrix A that is row equivalent to this matrix B in row echelon form:

$$A \xrightarrow{\text{Row ops}} B = \begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & \\ \hline 1 & 2 & 0 & 3 & 4 & 0 & 5 & 6 & 0 \\ 0 & 0 & 1 & 7 & 8 & 0 & 9 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 11 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{bmatrix} -3 \\ 0 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_2} + x_5 \underbrace{\begin{bmatrix} -4 \\ 0 \\ -8 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_3} + x_7 \underbrace{\begin{bmatrix} -5 \\ 0 \\ -9 \\ 0 \\ 0 \\ -11 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_4} + x_8 \underbrace{\begin{bmatrix} -6 \\ 0 \\ -10 \\ 0 \\ 0 \\ -12 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_5}$$

$$\text{Nul}(A) = \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\})$$

How is column dependence related?

The null space is exactly the list of all ways the colums are dependent.

4.2b: Column spaces MA322-001 Feb 26 Worksheet

Define Col(A) to be the span of the columns of A. It turns out this is always a subspace, no matter which matrix A is.

Verify the three part test to be a subspace here:

(a) Is $\vec{0}$ in Col(A)? In other words is $\vec{0}$ a linear combination of the columns of A?

Yes! $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + \cdots + 0\vec{v}_n = A\vec{0}$

(b) If $\vec{w}$ in Col(A), is $c\cdot\vec{w}$ in Col(A) for every c? In other words, if $\vec{w} = A\vec{x}$, is there a $\vec{y}$ with $c\cdot\vec{w} = A\vec{y}$? Yes. $\vec{y} = c\cdot\vec{x}$ since $A(c\cdot\vec{x}) = c(A\vec{x}) = c\cdot\vec{w}$

(c) If $\vec{v}, \vec{w}$ in Col(A), is $\vec{v}+\vec{w}$ in Col(A)? In other words, if $\vec{v} = A\vec{x}$ and $\vec{w} = A\vec{y}$, is there a $\vec{z}$ with $\vec{v}+\vec{w} = A\vec{z}$? Yes. $\vec{z} = \vec{x}+\vec{y}$ since $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{v}+\vec{w}$

It is easy to find some elements of Col(A) (the columns of A, any linear combination of them). However, given a vector $\vec{b}$, it can be hard to decide if $\vec{b}$ is in Col(A). How do we figure it out? RREF!

Explain how to test if a vector $\vec{b}$ all vectors in Col(A) for a matrix A that is row equivalent to this matrix B in row echelon form:

$$A \xrightarrow[R_3+R_1]{R_2-3R_1} \cdots \xrightarrow[R_1-R_4]{R_4-R3} B = \left[\begin{array}{cccccccc|c} 1 & 2 & 0 & 3 & 4 & 0 & 5 & 6 & 0 \\ 0 & 0 & 1 & 7 & 8 & 0 & 9 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 11 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$[A \mid \vec{b}] \longrightarrow [B \mid \vec{c}]$

Apply the row ops to $\vec{b}$ to get $\vec{c}$, check that $c_4 = 0$.

For example
$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-3R_1} \begin{bmatrix} b_1 \\ b_2-3b_1 \\ b_3+b_1 \\ b_4 \end{bmatrix} \xrightarrow[R_1-R_4]{R_4-R_3} \begin{bmatrix} b_1-b_4 \\ b_2-3b_1 \\ b_3+b_1 \\ b_4-b_3-b_1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

So we need $b_4 = b_1 + b_3$ for $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ to be in Col(A)

How is this related to row dependence?

The restrictions on $\vec{b}$ are exactly the dependencies of the rows of A. (A's 4th row is exactly the sum of its 1st and 3rd rows)

Matrices are convenient for computers and spreadsheets, but sometimes there are much clearer ways of describing a linear transformation.

A **linear transformation** is a function $T: V \to W$ between two vector spaces V and W (so $T(\vec{v}) = \vec{w}$) satisfying the following two axioms:

(Additive) $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$, and
(Multiple) $T(c\vec{v}) = cT(\vec{v})$

Let V be the vector space of polynomials and let T be the derivative. Is T a linear transformation? From where to where? Show the two part test:

$T: V \to V$, the derivative of a polynomial is a polynomial

(a) $T(\vec{v}_1 + \vec{v}_2) = \frac{d}{dt}(p(t) + q(t)) = p'(t) + q'(t) = T(\vec{v}_1) + T(\vec{v}_2)$

(b) $T(c\vec{v}_1) = \frac{d}{dt}(c \cdot p(t)) = c \cdot p'(t) = c \cdot T(\vec{v}_1)$

The **null space** of T is all $\vec{v}$ so that $T(\vec{v}) = \vec{0}$. If T is given by a matrix A, then the null space of T is just $\text{Nul}(A)$. What is the null space of T when T is the derivative operator?

$\text{Nul}(T) = \{\vec{v} : T(\vec{v}) = 0\} = \{p(t) : p'(t) = 0\}$

by calculus we know $p(t) = p(0) + \int_0^t p'(t)\,dt = p(0) + \int_0^t 0\,dt = p(0)$

that is, $p(t) = p(0)$ is a constant. $\text{Nul}(T) = \{a \in V : a \text{ is a number}\}$

The **image** of T is all $\vec{w}$ so that there is some $\vec{v}$ with $T(\vec{v}) = \vec{w}$. If T is given by a matrix A, the image of T is just $\text{Col}(A)$. What is the image of T when T is the derivative operator?

$\text{Im}(T) = \{T(\vec{v}) : \vec{v} \text{ in } V\} = \{p'(t) : p(t) \text{ is a polynomial}\}$

by calculus we know $f(t) = \frac{d}{dt}\left(\int_0^t f(x)\,dx\right)$

so set $p(t) = \int_0^t f(x)\,dx$. $p(t)$ is still a polynomial and

$T(p(t)) = p'(t) = f(t)$, so $f(t)$ is in $\text{Im}(T)$.

MA322-001 Feb 26 Quiz Name:____________________

4.1 (HW4.1#2) V is the vector space of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$, but W is only those vectors with $xy \geq 0$. For each of the three tests, check W:

(a) Is $\vec{0}$ in W? In other words is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ with $xy \geq 0$?

Yes! Clearly $x=0, y=0$ is the only solution, and $0 \cdot 0 = 0 \geq 0$

(b) If $\vec{v}$ in W and c is a number, is $c\vec{v}$ in W? In other words if $\begin{bmatrix} x \\ y \end{bmatrix}$ has $xy \geq 0$,

does $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ have $(cx)(cy) \geq 0$?

Yes! $(cx)(cy) = c^2 \cdot xy$ and both c^2, xy are ≥ 0

(c) if $\vec{v}$ and $\vec{u}$ are both in W, is $\vec{v}+\vec{u}$ in W? In other words if $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ have $xy \geq 0$ and $ab \geq 0$, does $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$ have $(x+a)(y+b) \geq 0$?

No! $(x+a)(y+b) = \underbrace{xy}_{\geq 0} + \underbrace{(bx+ay)}_{???} + \underbrace{ab}_{\geq 0}$. Take $x=1, y=0, a=0, b=-1$ then $\vec{u}+\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ but $1(-1) = -1 < 0$:(

4.2 (HW4.1#6) V is the vector space of all polynomials, $p(t)$. W is only those vectors of the form $p(t) = a + t^2$ for numbers a. Repeat the last question (parts a,b,c).

(a) Is $\vec{0}$ in W? Is there an a so that $a+t^2$ is the zero polynomial?
No! $a+t^2$ ~~is~~ always a parabola.

(b) t^2 in W, but $2 \cdot t^2$ is not in W :(

(c) t^2 and t^2 are in W, but t^2+t^2 is not in W :(

4.3 (HW4.1#19) V is the vector space of all rel valued functions, $f(t)$. W is only those that can be written as $f(t) = c_1 \cos(\pi t) + c_2 \sin(\pi t)$ for numbers c_1 and c_2. Repeat the last question (parts a,b,c)

(a) Yes, $c_1 = c_2 = 0$

(b) Yes, new $c_1 = c \cdot c_1$, new $c_2 = c \cdot c_2$

(c) Yes, new $c_1 = c_1 + d_1$, new $c_2 = c_2 + d_2$

too brief

I'll also look at your answers on the back (the derivative is linear, its null space is blank, its image is blank).