MA322-001 Mar 10 Review

1. Definition of vector space, subspace, and linear transformation.

(a) Set $V = \{p(t) : p \text{ is a quadratic polynomial}\}$. So $t^2 + 3t + 7$ and $-8t^2 + 5$ are in V, but $t - 3$ and $\sin(t)$ are not in V. Is V a vector space?

V is contained in the vector space of all polynomials, so most of the axioms come for free ($\vec{v} + \vec{u} = \vec{u} + \vec{v}$, etc.)

We only need the subspace check, but it fails all 3!

① $\vec{0} = 0t^2 + 0t + 0$ is not quadratic (notice $t-3$ not included)

② $0 \cdot (t^2 + 3t + 7) = \vec{0}$ is not quadratic

③ $(t^2 + 5t + 4) + (-t^2 - 4t - 7) = t - 3$ is not quadratic

(b) Set $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 2x + 3y = 5 \right\}$. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ are in U, but $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ are not in U. Is U a subspace of $\mathbb{R}^2$?

No. Fails all 3 checks!

① $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x = 0, y = 0$, but $2x + 3y = 2(0) + 3(0) = 0$, not 5

② $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in, but for $c = 2$, $c \cdot \vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is out. In general, if $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is in then $c \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ has $2(cx) + 3(cy) = c \cdot (2x + 3y) = 5c$ is only in if $c = 1$.

③ $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ are in, but $\vec{v} + \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ is not.

(c) Set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0 \right\}$. So $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$ are in W, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ \sqrt{2} \end{bmatrix}$ are not in W. Is W a subspace of $\mathbb{R}^2$?

No, but passes first two!

① $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x = 0$, $y = 0$, and $xy = 0(0) = 0$ ✓

② if $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in W, then $xy = 0$. if c is a number, then $c \cdot \vec{v} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ and $(cx)(cy) = c^2 xy = c^2(0) = 0$ ✓

☹ ③ but $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in W; while $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not ☹

$1(0) = 0$ ✓ $\quad 0(1) = 0$ ✓ $\quad 1(1) = 1 \neq 0$

Notation: $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \text{ are any numbers} \right\}$ is the vector space containing all column vectors of length 2. Many people think of it as the Euclidean plane.

2. Linear transformations (Make sure to justify your answers.)

(a) What are the two main requirements for a function to be a linear transformation?

$T: U \to W$ is a function from one vector space to another,

① $T(c \cdot \vec{u}) = c \cdot T(\vec{u})$ for every number c and vector $\vec{u}$ in U

② $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for every pair of vectors $\vec{u}, \vec{v}$ in U

(b) Set $T : \mathbb{R}^2 \to \mathbb{R}$ to be the function with $T\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = 2x + 3y$. So $T\left(\left[\begin{smallmatrix} \sqrt{2} \\ 7 \end{smallmatrix}\right]\right) = 2\sqrt{2} + 3(7) = 2\sqrt{2} + 21$. Is T a linear transformation?

Yes. (Both $\mathbb{R}^2$ and $\mathbb{R}$ are vector spaces.)

① $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $T(c \cdot \vec{u}) = T\left(c \begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = 2(cx) + 3(cy)$

$c \cdot T(\vec{u}) = c \cdot T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c(2x + 3y)$ (is the same as ↑, factor out the c)

② $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, $T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} x+a \\ y+b \end{bmatrix}\right) = 2(x+a) + 3(y+b)$

$T(\vec{u}) + T(\vec{v}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) + T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (2x + 3y) + (2a + 3b)$, just distribute

(c) Set $S : \mathbb{R}^2 \to \mathbb{R}$ to be the function with $S\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = xy$. So $S\left(\left[\begin{smallmatrix} \sqrt{2} \\ 7 \end{smallmatrix}\right]\right) = \sqrt{2}(7) = 7\sqrt{2}$. Is S a linear transformation?

No (Both $\mathbb{R}^2$ and $\mathbb{R}$ are vector spaces, but S fails the main two checks)

① $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $S(c \cdot \vec{u}) = S\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = (cx)(cy) = c^2 xy$

$c \cdot S(\vec{u}) = c \cdot S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c(xy)$. but $cxy \neq c^2 xy$ if $c \neq 1$

Take $c = x = y = 2$ for example, $8 \neq 16$.

② $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, $S(\vec{u} + \vec{v}) = S\left(\begin{bmatrix} x+a \\ y+b \end{bmatrix}\right) = (x+a)(y+b) = xy + (xb + ay) + ab$

$S(\vec{u}) + S(\vec{v}) = (xy) + (ab)$, but not equal if $xb + ay \neq 0$

Take $x = y = a = b = 1$ for example. $1 \cdot 1 + 1 \cdot 1 \neq (1+1) \cdot (1+1)$

$2 \neq 4$

3. Null spaces (Make sure to justify your answers.)

(a) Set $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -6 & 0 \end{bmatrix}$ and $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Is $\vec{\mathbf{x}}$ in the null space of A?

Is $A\vec{x} = \vec{0}$?

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & -6 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-1-2 \\ 6-6+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0} \checkmark$$

Yes.

(b) List 5 vectors from the null space of (that same) A.

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \\ -4 \end{bmatrix}$$

(c) Give a basis for the null space of (that same) A. Row Reduce A to find free vars.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & -6 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -1/2 & 0 \end{array}\right]$$

$x_1 = 3/2\, x_3$
$x_2 = 1/2\, x_3$
x_3 is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}, \text{ so } \left\{ \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix} \right\} \text{ is a basis}$$

$\left(\text{so is } \left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\} \right)$

(d) What is the dimension of the null space of (that same) A?

Dimension is the # of vectors in a basis, so

Dim = 1

(e) Give an example of a different matrix B that has the same null space as (that same) A.

$B = \begin{bmatrix} 2 & -6 & 0 \\ 1 & -1 & -1 \end{bmatrix}$, or $B = \begin{bmatrix} 4 & -12 & 0 \\ 2 & -2 & -2 \end{bmatrix}$, or even

$B = \begin{bmatrix} 1 & -1 & -1 \\ 6 & -4 & 2 \end{bmatrix}$. Row Ops Don't Change Null Space.

4. Column spaces (Make sure to justify your answers.)

(a) Set $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -6 & 0 \\ 1 & -3 & 0 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Is $\vec{\mathbf{b}}$ in the column space of A?

Is there $\vec{x}$ with $A\vec{x} = \vec{b}$?

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 2 & -6 & 0 & 1 \\ 1 & -3 & 0 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 0 & -4 & 2 & -5 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & -3/2 \end{array}\right] \not\approx$$ Last Row Cannot Be Solved

$\boxed{\text{No}}$ $\vec{b}$ is not in the column space.

(b) List 5 vectors from the column space of (that same) A. ($\vec{b}$ doesn't work, but $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ does)

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ -3 \end{bmatrix}, \underset{\text{Nice}}{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\text{Nice}}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

(c) Give a basis for the column space of (that same) A.

Standard Answer: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ -3 \end{bmatrix} \right\}$

(take columns of A that are pivots in $RREF(A)$)

Nice Answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

They are linearly independent, they are in the column space, there are the right number of them

(d) What is the dimension of the column space of (that same) A?

Dimension is size of a basis (any basis of the subspace), so

Dim = 2

(e) Give an example of another matrix C that has the same column space as (that same) A.

$$C = \begin{array}{c} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -6 \\ 0 & 1 & -3 \end{bmatrix} \\ \begin{array}{ccc} -v_3 & v_1 & v_2 \end{array} \end{array} \text{ or } C = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \\ \begin{array}{cc} -v_3 & v_1 - v_3 \end{array} \end{array},$$

"Column" ops don't change column space.

Can omit dependent columns.

I'll probably have A be the same matrix on pages 3 and 4, but the A from class (the one page 3) is not a good exam question on page 4.

wrong wrong

Note $C \neq \begin{bmatrix} 1 & -1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Row Ops Do Change Column Space.

wrong wrong