MA322-001 Mar 24 Worksheet - Eigenvalues and eigenvectors

An **eigenpair** of a matrix A is a vector $\vec{v}$ and a number c so that $A\vec{v} = c\vec{v}$. The vector $\vec{v}$ is called the **eigenvector** and the number c is called the **eigenvalue**.

Note that only square matrices can have eigenvectors. Why?

The input size (# of columns) must equal the output size (# of rows)

Eigenvectors are very handy: they change matrices into numbers.

Let's consider $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

What is $A\vec{v}$ in terms of $\vec{v}$?

$$A\vec{v} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ -2+4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\cdot\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{2\vec{v}}$$

What is $A\vec{w}$ in terms of $\vec{w}$?

$$A\vec{w} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\cdot\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{3\vec{w}}$$

What is $A(17\vec{v} + 23\sqrt{2}\vec{w})$ in terms of $\vec{v}$ and $\vec{w}$?

$$17(A\vec{v}) + (23\sqrt{2})(A\vec{w}) = 17(2\vec{v}) + (23\sqrt{2})(3\vec{w})$$
$$= 34\vec{v} + 69\sqrt{2}\vec{w}$$

What is $A(x\vec{v} + y\vec{w})$? Call it $b\vec{v} + c\vec{w}$.

$$x(A\vec{v}) + y(A\vec{w}) = x(2\vec{v}) + y(3\vec{w}) = (2x)\vec{v} + (3y)\vec{w}$$

$b = 2x$, $c = 3y$

What matrix takes $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ to $\vec{b} = \begin{bmatrix} b \\ c \end{bmatrix}$?

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

To find ? remember $\tilde{A} = [\tilde{A}e_1 \mid \tilde{A}e_2]$, $\tilde{A}e_1 = \begin{bmatrix} 2\cdot 1 \\ 3\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\tilde{A}e_2 = \begin{bmatrix} 2\cdot 0 \\ 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

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(5.1a) What are the eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$?

Eigenvalues are #s that the matrix acts like.

$\begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x \\ -\sqrt{2}y \end{bmatrix}$, it multiplies x by 1 and y by $-\sqrt{2}$

$c = 1$, $c = -\sqrt{2}$

(5.1b) What are the eigenvectors of $A = \begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$?

"The x and the y"

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ since $\begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c = 1$

$\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ since $\begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} = -\sqrt{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = \sqrt{2}$

(5.1c) Give a different matrix with the same eigenvalues:

(Tricky)

$\begin{bmatrix} -\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$ works. Just swap x and y

(5.1d) Give a different matrix with the same eigenvectors:

(Tricky)

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for any a, b works

(5.1e) What are the eigenvalues of $A^3 + 3A + 5I$?

$A^3 = A \cdot A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1^2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1^3 & 0 \\ 0 & (-\sqrt{2})^3 \end{bmatrix}$

$A^3 + 3A + 5I = \begin{bmatrix} 1^3 + 3(1) + 5 & 0 \\ 0 & (-\sqrt{2})^3 + 3(-\sqrt{2}) + 5 \end{bmatrix}$

$c = 1^3 + 3(1) + 5$, $c = (-\sqrt{2})^3 + 3(-\sqrt{2}) + 5$