MA322-001 Mar 26 Worksheet - Diagonal matrices (like 5.3)

A **diagonal matrix** can be defined as a matrix with eigenvectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. The corresponding eigenvalues $c_1, c_2, \ldots, c_n$ are called its diagonal entries. For example, the diagonal matrix with diagonal entries $4, 7, 13$ is the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

What is $A\vec{e}_2$ in terms of $\vec{e}_2$?

Arithmetic with diagonal matrices is a little easier than with general matrices. What is A^2?

$$A^2 = A \cdot A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 4(4)+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+7(7)+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+13(13) \end{bmatrix} = \begin{bmatrix} 4^2 & 0 & 0 \\ 0 & 7^2 & 0 \\ 0 & 0 & 13^2 \end{bmatrix}$$

What is A^3?

$$A^3 = A \cdot A^2 \quad \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} 4^2 & 0 & 0 \\ 0 & 7^2 & 0 \\ 0 & 0 & 13^2 \end{bmatrix} = \begin{bmatrix} 4^3 & 0 & 0 \\ 0 & 7^3 & 0 \\ 0 & 0 & 13^3 \end{bmatrix}$$

What is A^{10}? (No need to simplify.)

$$\begin{bmatrix} 4^{10} & 0 & 0 \\ 0 & 7^{10} & 0 \\ 0 & 0 & 13^{10} \end{bmatrix}$$

What is $A^5 + 11A^2 + 17A + 5I$? (You don't need to simplify.)

$$\begin{bmatrix} 4^5 + 11(4)^2 + 17(4) + 5 & 0 & 0 \\ 0 & 7^5 + 11(7)^2 + 17(7) + 5 & 0 \\ 0 & 0 & 13^5 + 11(13)^2 + 17(13) + 5 \end{bmatrix}$$

If you know Taylor series or are good with patterns, what do you think $\sin(A)$ is?

$$\sin(A) = A - A^3/6 + A^5/120 - + \cdots \qquad \text{so} \quad \text{same trick works}$$

$$\sin(A) = \begin{bmatrix} \sin(4) & 0 & 0 \\ 0 & \sin(7) & 0 \\ 0 & 0 & \sin(13) \end{bmatrix}$$

Let B be the diagonal matrix with diagonal entries $\sqrt{2}, \sqrt{3}, \sqrt{5}$. What is AB?

$$AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 4\sqrt{2}+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+7\sqrt{3}+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+13\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} & 0 & 0 \\ 0 & 7\sqrt{3} & 0 \\ 0 & 0 & 13\sqrt{5} \end{bmatrix}$$

What is BA?

$$BA = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix} = \begin{bmatrix} \sqrt{2}(4) & 0 & 0 \\ 0 & \sqrt{3}(7) & 0 \\ 0 & 0 & \sqrt{5}(13) \end{bmatrix} = AB$$

What is $A + B$?

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix} + \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 4+\sqrt{2} & 0+0 & 0+0 \\ 0+0 & 7+\sqrt{3} & 0+0 \\ 0+0 & 0+0 & 13+\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4+\sqrt{2} & 0 & 0 \\ 0 & 7+\sqrt{3} & 0 \\ 0 & 0 & 13+\sqrt{5} \end{bmatrix}$$

MA322-001 Mar 26 Worksheet - 5.2 Finding eigenvalues

Finding eigenvectors given an eigenvalue is just solving $A\vec{x} = c\vec{x}$. This is the same as solving $(A - cI)\vec{x} = \vec{0}$, a null space calculation.

Find the eigenvector of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ associated to the eigenvalue $c = 2$.

$$A\vec{x} = c\vec{x}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 = 2x_1 \\ 2x_2 = 2x_2 \end{cases}$$

$$\begin{cases} x_1 = -x_2 \\ x_2 \text{ is free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ 1 \cdot x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ is an eigenvector for every } x_2 \neq 0$$

take $x_2 = 1$ for instance

$$\left(2, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

Finding eigenvalues given only the matrix is hard in general. However, for up to 4×4 we can use quadratic (or cubic, or quartic) formulas to find the eigenvalues. The **characteristic polynomial** of a matrix is defined to be

$$\det(A - xI)$$

and is an nth degree polynomial if A is $n \times n$. The zeros of the characteristic polynomial are exactly the eigenvalues of A. In particular, if $n = 2$, then we just use quadratic formula to find the eigenvalues.

Find the egienvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

$$\det(A - cI) = (1-c)(2-c) - 1(0)$$
$$= 2 - 3c + c^2 \cdots$$
$$= (1-c)(2-c)$$

has roots $c = 1$ and $c = 2$

$$A\vec{x} = c\vec{x}$$

$$(A - cI)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \right)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1-c & 1 \\ 0 & 2-c \end{bmatrix}\vec{x} = \vec{0}$$

has a nontrivial solution iff $\det\left(\begin{bmatrix} 1-c & 1 \\ 0 & 2-c \end{bmatrix} \right) = 0$

Why does the characteristic polynomial trick work?

Reality note: If n is larger than about 10, then root finding algorithms are not very well suited. For the same cost as calculating the polynomial (about n^3 operations), we can get a fairly accurate estimate not only of the eigenvalue (nearest to any chosen number) but also the eigenvector that goes with it. See page 322 in section 5.8 for the easiest and fairly effective method (inverse iteration). In fact, for the same cost, we can find good approximations to the eigenvalues closest to any set of n numbers, and the approximations can be improved very quickly (QR iteration).

Fibonacci via eigenvalues

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. What are its eigenpairs?

$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\right) = (-c)(1-c) - 1 = c^2 - c - 1$

has roots $c = \dfrac{-(-1) \pm \sqrt{(-1)^2 - (4)(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$

$A - c = \begin{bmatrix} -c & 1 \\ 1 & 1-c \end{bmatrix} \xrightarrow{R_2 \cdot (c) + R_1} \begin{bmatrix} -c & 1 \\ 0 & c - c^2 + 1 \end{bmatrix} = \begin{bmatrix} -c & 1 \\ 0 & 0 \end{bmatrix}$ so $X_2 = c \cdot X_1$

$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1 \begin{bmatrix} 1 \\ c \end{bmatrix}$

$\left(c, \begin{bmatrix} 1 \\ c \end{bmatrix} \right)$

Write the vector $\vec{e}_2$ as a linear combination of your eigenvectors.
(Hint: it is probably $\pm(\vec{v}_1 - \vec{v}_2)/\sqrt{5}$.)

$\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1-1 \\ \sqrt{5} \end{bmatrix} = \sqrt{5} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Consider the sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots, x_k, \ldots$ with $x_{k+2} = x_{k+1} + x_k$. What is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$?

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_k \\ X_{k+1} \end{bmatrix} = \begin{bmatrix} X_{k+1} \\ X_k + X_{k+1} \end{bmatrix} = \begin{bmatrix} X_{k+1} \\ X_{k+2} \end{bmatrix}$

Suppose we start with $\vec{e}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. What do you get from $A^k \vec{e}_2$?

$A\vec{e}_2 = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix}, \quad A^2\vec{e}_2 = \begin{bmatrix} X_3 \\ X_4 \end{bmatrix}, \quad A^k \vec{e}_2 = \begin{bmatrix} X_{k+1} \\ X_{k+2} \end{bmatrix}$

Suppose we start with a linear combination of your eigenvectors $a_1\vec{v}_1 + a_2\vec{v}_2$. What do we get from $A^k(a_1\vec{v}_1 + a_2\vec{v}_2)$?

$A^k\left(a_1\vec{v}_1 + a_2\vec{v}_2\right) = a_1\left(A^k\vec{v}_1\right) + a_2\left(A^k\vec{v}_2\right) = a_1 c_1^k \vec{v}_1 + a_2 c_2^k \vec{v}_2$

One of your eigenvalues should be about 0.6. What is $(0.6)^{35}$?

≈ 0

≈ 5702886.91
$\approx \boxed{5702887}$

Approximately what is x_{35}? Can you find it exactly, knowing it has to be an integer?

$\begin{bmatrix} X_{35} \\ X_{36} \end{bmatrix} = A^{34}\vec{e}_2 = A^{34}\left(\dfrac{\vec{v}_1 - \vec{v}_2}{\sqrt{5}}\right) = \dfrac{c_1^{34}}{\sqrt{5}}\vec{v}_1 - \dfrac{c_2^{34}}{\sqrt{5}}\vec{v}_2$

$\approx \dfrac{c_1^{34}}{\sqrt{5}} \begin{bmatrix} 1 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_1^{34}/\sqrt{5} \\ c_1^{35}/\sqrt{5} \end{bmatrix} = \boxed{\left(\dfrac{1+\sqrt{5}}{2}\right)^{34} / \sqrt{5}}$

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(5.1a) Is $c = 2$ an eigenvalue of $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

$$\det\left(\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) = 1(6) - 2(3) = 0$$

So yes.

In fact $\begin{bmatrix} 1 & 2 & | & 0 \\ 3 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{matrix} x_1 = -2x_2 \\ x_2 = \text{free} \end{matrix} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\left(2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$ is an eigenpair

(5.1b) Is $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $B = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$?

$$B\vec{v} = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1-3 \\ 6-12 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c\vec{v}$$

Yes. In fact, $\left(-2, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$ is an eigen pair

(5.1c) Find the eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}\right) = \det\left(\begin{bmatrix} -c & 0 & 0 \\ 0 & 3-c & 4 \\ 0 & 0 & -2-c \end{bmatrix}\right)$$

$= (-c)(3-c)(-2-c)$ is 0 if and only if

$\boxed{c = 0 \quad \text{or} \quad c = 3 \quad \text{or} \quad c = -2}$ (the diagonal entries

of any upper triangular matrix)