MA322-001 Apr 2 Exam Name: Key

1. Eigenvectors

(a) Give the definition of eigenvector. A vector $\vec{x}$ is an eigenvector of the Matrix A if there is a number c so that

$A\vec{x} = c\vec{x}$ and $\vec{x} \neq 0$

(b) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 9 & 3 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix}$? Explain why or why not.

$$\begin{bmatrix} 9 & 3 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9+6-12 \\ 12-6 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(c) Why do only square matrices have eigenvectors? if A is $m \times n$

$\vec{x}$ is $n \times 1$

$A\vec{x}$ is $m \times 1$

unless $m = n$, $A\vec{x}$ cannot be a scalar multiple of $\vec{x}$

(d) Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 9 & 3 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix}$? Explain why or why not.

$$\begin{bmatrix} 9 & 3 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} \neq c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

2. Eigenvalues:

(a) Give the definition of eigenvalue. A number c is an eigenvalue of the matrix A if there is a nonzero vector $\vec{x}$ so that $A\vec{x} = c\vec{x}$

(b) Is -4 an eigenvalue of $\begin{bmatrix} 9 & 3 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix}$? Explain why or why not.

No $\begin{bmatrix} 9 & 3 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 13 & 3 & -4 \\ 0 & 10 & -2 \\ 0 & 0 & 7 \end{bmatrix}$

is invertible (already in REF) so $A\vec{x} = (-4)\vec{x}$ implies $\vec{x} = 0$

(c) Is 9 an eigenvalue of $\begin{bmatrix} 9 & 3 & -4 \\ 0 & 6 & -2 \\ 0 & 0 & 3 \end{bmatrix}$? Explain why or why not.

Yes $\left[\begin{array}{ccc|c} 0 & 3 & -4 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ has $x_1 = $ free, $x_2 = 0$, $x_3 = 0$

so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an e-vector for $c = 9$

(d) Give an example of a matrix with eigenvalue 5. Explain why your example works.

$[5]$

3. Diagonal matrices

On this page, $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $D = \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix}$, and $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(a) Calculate $A^7 - 5A^2 + A$

$$\begin{bmatrix} a^7 - 5a^2 + a & 0 & 0 \\ 0 & b^7 - 5b^2 + b & 0 \\ 0 & 0 & c^7 - 5c^2 + c \end{bmatrix}$$

(b) Calculate $(A + D)^2$

$$\begin{bmatrix} (a+d)^2 & 0 & 0 \\ 0 & (b+e)^2 & 0 \\ 0 & 0 & (c+f)^2 \end{bmatrix}$$

(c) Calculate $(D^5 - 7D^3)\vec{\mathbf{v}}$

$$\begin{bmatrix} (d^5 - 7d^3)x \\ (e^5 - 7e^3)y \\ (f^5 - 7f^3)z \end{bmatrix}$$

(d) What are the eigenpairs of $A + D$?

$$\left(a+d, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(b+e, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \left(c+f, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

4. General matrices

On this page, $A = \begin{bmatrix} 5 & 0 & -4 \\ 1 & 2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$, and it has

eigenpairs $(c_1, \vec{\mathbf{v}}_1) = \left(1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$, $(c_2, \vec{\mathbf{v}}_2) = \left(2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$, and $(c_3, \vec{\mathbf{v}}_3) = \left(3, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right)$.

The vector $\vec{\mathbf{w}} = \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix}$ and it can be written as $4\vec{\mathbf{v}}_1 + 5\vec{\mathbf{v}}_2$.

(a) Calculate $(A^7 - 5A^2 + A)\vec{\mathbf{v}}_1$

$$\left(1^7 - 5(1)^2 + 1\right)\vec{v}_1 = \begin{bmatrix} (1^7 - 5(1)^2 + 1)\,1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}$$

(b) Calculate $(A^7 - 5A^2 + A)\vec{\mathbf{v}}_2$

$$\left(2^7 - 5(2)^2 + 2\right)\vec{V}_2 = \begin{bmatrix} 0 \\ 128 - 20 + 2 = 110 \\ 0 \end{bmatrix}$$

(c) Calculate $(A^7 - 5A^2 + A)\vec{\mathbf{w}}$

$$(-3)(4\vec{v}_1) + (110)(5\vec{v}_2) = \begin{bmatrix} -12 \\ 550 \\ -12 \end{bmatrix}$$

(d) What are the eigenpairs of A^2?

$$\left(1^2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right), \left(2^2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \left(3^2, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right)$$