MA322-001 Apr 7 Cliff Notes - Inner products

An **inner product** is a function that takes two vectors and gives a scalar, $\langle \vec{v}, \vec{w} \rangle = c$ or $\vec{v} \cdot \vec{w}$. It must satisfy the following two (familiar) axioms on addition and multiplication:

(A_1)	$\langle \vec{v}_1 + \vec{v}_2, \vec{w} \rangle = \langle \vec{v}_1, \vec{w} \rangle + \langle \vec{v}_2, \vec{w} \rangle$	$(\vec{v}_1 + \vec{v}_2) \cdot \vec{w} = \vec{v}_1 \cdot \vec{w} + \vec{v}_2 \cdot \vec{w}$	(addition works)
(A_2)	$\langle \vec{v}, \vec{w}_1 + \vec{w}_2 \rangle = \langle \vec{v}, \vec{w}_1 \rangle + \langle \vec{v}, \vec{w}_2 \rangle$	$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2$	(on both sides)
(M_1)	$\langle c\vec{v}, \vec{w} \rangle = c\langle \vec{v}, \vec{w} \rangle$	$(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$	(scalars work)
(M_2)	$\langle \vec{v}, c\vec{w} \rangle = c\langle \vec{v}, \vec{w} \rangle$	$\vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$	(on both sides)

Examples of inner products:

$L^2(\mathbb{R})$ • If vectors are real-valued integrable functions, like $\vec{v} = f(x)$ and $\vec{w} = g(x)$, then

$$\langle \vec{v}, \vec{w} \rangle = \int_{-\infty}^{\infty} f(x)g(x)\ dx$$

is an inner product.

$\mathbb{R}^n$ • If vectors are lists of numbers, like $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$, then

$$\langle \vec{v}, \vec{w} \rangle = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

is an inner product.

A **length** or **norm** is a function that takes a vector and gives a scalar (and that satisfies several less familiar axioms similar to the absolute value $|x|$). We'll only use one norm, the 2-norm, and it is defined:

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\int_{-\infty}^{\infty} f(x)^2\ dx} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

In $\mathbb{R}^n$ we get a very important rule called the Law of Cosines:

$$\begin{array}{rclclcl} c^2 & = & a^2 & + & b^2 & + & 2ab\cos(\theta) \\ \|\vec{v} + \vec{w}\|^2 & = & \|\vec{v}\|^2 & + & \|\vec{w}\|^2 & + & 2\|\vec{v}\|\|\vec{w}\|\cos(\theta) \\ \langle \vec{v} + \vec{w}, \vec{v} + \vec{w} \rangle & = & \langle \vec{v}, \vec{v} \rangle & + & \langle \vec{w}, \vec{w} \rangle & + & 2\langle \vec{v}, \vec{w} \rangle \end{array}$$

or more simply:

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\|\|\vec{w}\|\cos(\theta), \qquad \theta \text{ is the angle between the vectors}$$

Two vectors are called **orthogonal** if the angle between them is 90^o, that is, if $\cos(\theta) = 0$, that is, if $\langle \vec{v}, \vec{w} \rangle = 0$.

Examples of orthogonal vectors:

$\mathbb{R}^n$ • the standard basis vectors (x/y/z directions; $\mathbf{i}, \mathbf{j}, \mathbf{k}$; $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$)

$L^2([0, 2\pi])$ • the Fourier basis $\sin(n\theta)$, $\cos(n\theta)$

Eigen • if A is a linear transformation satisfying $\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A\vec{w} \rangle$, then eigenvectors for distinct eigenvalues are orthogonal.

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Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

1. Compute the following inner-products for $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

(a) $\vec{v} \cdot \vec{e}_1$

(b) $\vec{v} \cdot \vec{e}_2$

(c) $\vec{v} \cdot \vec{e}_3$

Let $\vec{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{f}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{f}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

2. Compute the following inner-products for $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

(a) $\vec{v} \cdot \vec{f}_1$

(b) $\vec{v} \cdot \vec{f}_2$

(c) $\vec{v} \cdot \vec{f}_3$

Let $\vec{g}_1 \cdot \vec{g}_2 = 0$ and $\|\vec{g}_1\| = \|\vec{g}_2\| = 1$.

3. Compute the following inner products for $\vec{v} = 4\vec{g}_1 + 5\vec{g}_2$

(a) $\vec{v} \cdot \vec{g}_1$

(b) $\vec{v} \cdot \vec{g}_2$

4. What is the lesson?