MA322-001 Apr 9 Cliff Notes - Projections

The **projection** of a vector $\vec{v}$ onto a vector $\vec{w}$ is the multiple of $\vec{w}$ that is nearest to $\vec{v}$.

Calculus interlude: The multiples of $\vec{w}$ are $t\vec{w}$, so which value of t is best? Let $f(t) = \|\vec{v} - t\vec{w}\|$. Then

$$f(t)^2 = \langle \vec{v} - t\vec{w}, \vec{v} - t\vec{w} \rangle = \langle \vec{v}, \vec{v} \rangle - 2\langle \vec{v}, \vec{w} \rangle t + \langle \vec{w}, \vec{w} \rangle t^2$$

is quadratic, so its minimum (and the minimum of $f(t)$) occurs at "$-\frac{b}{2a}$", that is at $t = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$.

The **formula** for the projection of $\vec{v}$ onto $\vec{w}$ is thus

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}$$

Example: Define $\vec{g}_1 = \begin{bmatrix} 4/9 \\ 4/9 \\ 7/9 \end{bmatrix}$, $\vec{g}_2 = \begin{bmatrix} 1/9 \\ -8/9 \\ 4/9 \end{bmatrix}$, $\vec{g}_3 = \begin{bmatrix} 8/9 \\ -1/9 \\ -4/9 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$.

By the way, $\vec{g}_i \cdot \vec{g}_j = 0$ if $i \neq j$ and $\vec{g}_i \cdot \vec{g}_i = 1$.

Find the projection of $\vec{v}$ onto $\vec{g}_1$: $7\vec{g}_1 = \frac{1}{9}\begin{bmatrix} 28 \\ 28 \\ 49 \end{bmatrix}$

$\vec{g}_1 \cdot \vec{g}_1 =?$ 1

$\vec{v} \cdot \vec{g}_1 =?$ 7

Find the projection of $\vec{v}$ onto $\vec{g}_2$: $-3\vec{g}_2 = \frac{1}{9}\begin{bmatrix} -3 \\ 24 \\ -12 \end{bmatrix}$

$\vec{g}_2 \cdot \vec{g}_2 =?$ 1

$\vec{v} \cdot \vec{g}_2 =?$ -3

Find the projection of $\vec{v}$ onto $\vec{g}_3$: $-2\vec{g}_3 = \frac{1}{9}\begin{bmatrix} -16 \\ 2 \\ 8 \end{bmatrix}$

$\vec{g}_3 \cdot \vec{g}_3 =?$ 1

$\vec{v} \cdot \vec{g}_3 =?$ -2

What happens when you add them up?

$$= \frac{1}{9}\begin{bmatrix} 28-3-16 \\ 28+24+2 \\ 49-12+8 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 54 \\ 45 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix} = \vec{v}$$

Find x_1, x_2, and x_3 such that $\vec{v} = x_1\vec{g}_1 + x_2\vec{g}_2 + x_3\vec{g}_3$.

$7\vec{g}_1 - 3\vec{g}_2 - 2\vec{g}_3$

$X_1 = 7, \quad X_2 = -3, \quad X_3 = -2$

How does this magic work? Well, define the matrix $G = \begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \end{bmatrix} = \begin{bmatrix} 4/9 & 1/9 & 8/9 \\ 4/9 & -8/9 & -1/9 \\ 7/9 & 4/9 & -4/9 \end{bmatrix}$.

How is G^TG related to $\langle \vec{g}_i, \vec{g}_j \rangle$?

this is the (i,j)th entry of G^TG

so $G^TG = I$

So that means $G^{-1} = G^T$. Hence multiplying by G^T solves systems of equations, like $G\vec{x} = \vec{v}$.

MA322-001 Apr 9 Quiz Name: Key

Let $\vec{g}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{g}_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$.

1. Compute the following:

(a) $\langle \vec{g}_1, \vec{g}_1 \rangle$ $(3/5)(3/5) + (4/5)(4/5) = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$

(b) $\langle \vec{v}, \vec{g}_1 \rangle$ $(7)(3/5) + (11)(4/5) = \frac{21}{5} + \frac{44}{5} = \frac{65}{5} = 13$

(c) The projection $\vec{u}$ of $\vec{v}$ onto $\vec{g}_1$: $\vec{u} = \frac{\vec{v} \cdot \vec{g}_1}{g_1 \cdot g_1} g_1 = \frac{13}{1} g_1 = \begin{bmatrix} 39/5 \\ 52/5 \end{bmatrix}$

(d) $\langle \vec{u}, \vec{g}_2 \rangle$ $(39/5)(-4/5) + (52/5)(3/5) = -\frac{156}{25} + \frac{156}{25} = 0$

2. Compute the following:

(a) $\langle \vec{g}_2, \vec{g}_2 \rangle$ $(-4/5)(-4/5) + (3/5)(3/5) = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$

(b) $\langle \vec{v}, \vec{g}_2 \rangle$ $(7)(-4/5) + (11)(3/5) = \frac{-28}{5} + \frac{33}{5} = \frac{5}{5} = 1$

(c) The projection $\vec{w}$ of $\vec{v}$ onto $\vec{g}_2$: $\vec{w} = \frac{\vec{v} \cdot \vec{g}_2}{g_2 \cdot g_2} \vec{g}_2 = \frac{1}{1} g_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$

(d) $\langle \vec{w}, \cancel{\vec{g}_2} \rangle$ $\langle \vec{w}, \vec{g}_1 \rangle = (-4/5)(3/5) + (3/5)(\frac{4}{5})$
$= \frac{-12}{5} + \frac{12}{5} = 0$

3. What is $\vec{u} + \vec{w}$? $\begin{bmatrix} 39/5 \\ 52/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 35/5 \\ 55/5 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \vec{v}$!

4. Find numbers x_1 and x_2 so that $\vec{v} = x_1\vec{g}_1 + x_2\vec{g}_2$.

$\vec{v} = \vec{u} + \vec{w}$

$\vec{v} = 13\vec{g}_1 + 1\vec{g}_2$

$x_1 = 13, \quad x_2 = 1$