MA322-001 Apr 9 Cliff Notes - Projections

The **projection** of a vector $\vec{v}$ onto a vector $\vec{w}$ is the multiple of $\vec{w}$ that is nearest to $\vec{v}$.

Calculus interlude: The multiples of $\vec{w}$ are $t\vec{w}$, so which value of t is best? Let $f(t) = \|\vec{v} - t\vec{w}\|$. Then

$$f(t)^2 = \langle \vec{v} - t\vec{w}, \vec{v} - t\vec{w} \rangle = \langle \vec{v}, \vec{v} \rangle - 2\langle \vec{v}, \vec{w} \rangle t + \langle \vec{w}, \vec{w} \rangle t^2$$

is quadratic, so its minimum (and the minimum of $f(t)$) occurs at "$-\frac{b}{2a}$", that is at $t = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$.

The **formula** for the projection of $\vec{v}$ onto $\vec{w}$ is thus

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$$

Example: Define $\vec{g}_1 = \begin{bmatrix} 4/9 \\ 4/9 \\ 7/9 \end{bmatrix}$, $\vec{g}_2 = \begin{bmatrix} 1/9 \\ -8/9 \\ 4/9 \end{bmatrix}$, $\vec{g}_3 = \begin{bmatrix} 8/9 \\ -1/9 \\ -4/9 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$.

By the way, $\vec{g}_i \cdot \vec{g}_j = 0$ if $i \neq j$ and $\vec{g}_i \cdot \vec{g}_i = 1$.

Find the projection of $\vec{v}$ onto $\vec{g}_1$:
$\vec{g}_1 \cdot \vec{g}_1 =?$
$\vec{v} \cdot \vec{g}_1 =?$

Find the projection of $\vec{v}$ onto $\vec{g}_2$:
$\vec{g}_2 \cdot \vec{g}_2 =?$
$\vec{v} \cdot \vec{g}_2 =?$

Find the projection of $\vec{v}$ onto $\vec{g}_3$:
$\vec{g}_3 \cdot \vec{g}_3 =?$
$\vec{v} \cdot \vec{g}_3 =?$

What happens when you add them up?

Find x_1, x_2, and x_3 such that $\vec{v} = x_1\vec{g}_1 + x_2\vec{g}_2 + x_3\vec{g}_3$.

How does this magic work? Well, define the matrix $G = \begin{bmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \end{bmatrix} = \begin{bmatrix} 4/9 & 1/9 & 8/9 \\ 4/9 & -8/9 & -1/9 \\ 7/9 & 4/9 & -4/9 \end{bmatrix}$.

How is $G^T G$ related to $\langle \vec{g}_i, \vec{g}_j \rangle$?

So that means $G^{-1} = G^T$. Hence multiplying by G^T solves systems of equations, like $G\vec{x} = \vec{v}$.

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Let $\vec{g}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, $\vec{g}_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$, and $\vec{v} = \left[\begin{smallmatrix} 7 \\ 11 \end{smallmatrix}\right]$.

1. Compute the following:

(a) $\langle \vec{g}_1, \vec{g}_1 \rangle$

(b) $\langle \vec{v}, \vec{g}_1 \rangle$

(c) The projection $\vec{u}$ of $\vec{v}$ onto $\vec{g}_1$:

(d) $\langle \vec{u}, \vec{g}_2 \rangle$

2. Compute the following:

(a) $\langle \vec{g}_2, \vec{g}_2 \rangle$

(b) $\langle \vec{v}, \vec{g}_2 \rangle$

(c) The projection $\vec{w}$ of $\vec{v}$ onto $\vec{g}_2$:

(d) $\langle \vec{w}, \vec{g}_2 \rangle$

3. What is $\vec{u} + \vec{w}$?

4. Find numbers x_1 and x_2 so that $\vec{v} = x_1\vec{g}_1 + x_2\vec{g}_2$.