MA322-001 Apr 11 Cliff Notes - Properties of orthogonal projection

Let $\vec{\mathbf{g}}_1$, $\vec{\mathbf{g}}_2$, $\vec{\mathbf{g}}_3$ be vectors in a vector space with $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ if $i \neq j$ and with $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i \neq 0$.
Set $\vec{\mathbf{v}} = x_1\vec{\mathbf{g}}_1 + x_2\vec{\mathbf{g}}_2 + x_3\vec{\mathbf{g}}_3$ and find an expression for $\|\vec{\mathbf{v}}\|^2 = \|x_1\vec{\mathbf{g}}_1 + x_2\vec{\mathbf{g}}_2 + x_3\vec{\mathbf{g}}_3\|^2$

$$\|\vec{v}\|^2 = (x_1\vec{g}_1 + x_2\vec{g}_2 + x_3\vec{g}_3)\cdot(x_1\vec{g}_1 + x_2\vec{g}_2 + x_3\vec{g}_3) = x_1^2(\vec{g}_1\cdot\vec{g}_1) + x_2^2(\vec{g}_2\cdot\vec{g}_2) + x_3^2(\vec{g}_3\cdot\vec{g}_3) + 2x_1x_2(\vec{g}_1\cdot\vec{g}_2) + 2x_2x_3(\vec{g}_2\cdot\vec{g}_3) + 2x_3x_1(\vec{g}_3\cdot\vec{g}_1)$$

$$= x_1^2\|\vec{g}_1\|^2 + x_2\|\vec{g}_2\|^2 + x_3\|\vec{g}_3\|^2$$

Find an expression for $\vec{\mathbf{v}} \cdot \vec{\mathbf{g}}_i$

$$\vec{v}\cdot\vec{g}_i = (x_1\vec{g}_1 + x_2\vec{g}_2 + x_3\vec{g}_3)\cdot\vec{g}_i$$

$$= x_i\,\vec{g}_i\cdot\vec{g}_i + \cancel{x_j\,\vec{g}_j\cdot\vec{g}_i}^{\,0} = x_i\|g_i\|^2$$

Do you need $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ for $i \neq j$?

Yes! All the 0 are because of that

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ be vectors in a vector space where $\langle\vec{\mathbf{v}}, \vec{\mathbf{v}}\rangle \neq 0$ and $\langle\vec{\mathbf{a}}, \vec{\mathbf{b}}\rangle = \langle\vec{\mathbf{b}}, \vec{\mathbf{a}}\rangle$.

Define $\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{w}}}{\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}}\vec{\mathbf{v}}$.

We have the following properties:

- $\vec{\mathbf{w}} \mapsto \text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}})$ is linear
- $\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \text{proj}_{\vec{\mathbf{v}}}(\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}))$
- If $\vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{a}} \cdot \text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = 0$
- If $\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} = 0$, then $\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) \cdot \vec{\mathbf{a}} = 0$
- $\vec{\mathbf{v}} \cdot \text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$
- $\vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} - \text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}})) = 0$

Just check these to get used to dot products and vectors and dot products, oh my!

Gram-Schmidt: Replace a sequence of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots$ with a sequence of orthogonal vectors $\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \ldots$ with the same span.

For $i = 1, 2, \ldots$
  Set $\vec{\mathbf{g}}_i = \vec{\mathbf{v}}_i$
  For $j = 1, 2, \ldots, i-1$
    Set $r_{ij} = \frac{\vec{\mathbf{g}}_j\cdot\vec{\mathbf{g}}_i}{r_{jj}}$ if $r_{jj} \neq 0$
    Set $\vec{\mathbf{g}}_i = \vec{\mathbf{g}}_i - r_{ij}\vec{\mathbf{g}}_j$
  End For
  Set $r_{ii} = \vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i$
End For
For $i = 1, 2, \ldots$
  Set $r_{ii} = 1$
End For

Rational modified (num. stable) Gram-Schmidt algorithm for arbitrary bilinear form.
r should be initialized to 0

At the end of the ith iteration, $\vec{\mathbf{v}}_i = r_{i1}\vec{\mathbf{g}}_1 + \ldots + r_{(i-1),1}\vec{\mathbf{g}}_{i-1} + \vec{\mathbf{g}}_i$ and $\vec{\mathbf{g}}_j \cdot \vec{\mathbf{g}}_k = 0$ for all $1 \leq j < k \leq i$. If $r_{ii} = 0$, then $\vec{\mathbf{v}}_i$ is linearly dependent on the previous $\vec{\mathbf{v}}_j$, so $\vec{\mathbf{g}}_i = \vec{\mathbf{0}}$ could be omitted (or replaced with a random vector orthogonal to the previous vectors).

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Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}$.

1. Find vectors $\vec{g}_1$ and $\vec{g}_2$ with the same span as $\vec{v}_1$ and $\vec{v}_2$, except with $\vec{g}_1 \cdot \vec{g}_2 = 0$.

$$\vec{g}_1 = \vec{v}_1$$

$$\vec{g}_2 = \vec{v}_2 - \text{proj}_{\vec{g}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1 = \vec{v}_2 - \frac{3+12}{9+16}\vec{g}_1 = \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}} - \frac{3}{5}\overset{\vec{g}_1}{\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix}$$

2. What is the projection of $\vec{b}$ onto $\vec{g}_1$

$$\text{proj}_{\vec{g}_1}(\vec{b}) = \frac{\vec{b} \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1 = \frac{21+44}{25}\vec{g}_1 = \frac{65}{25} = \frac{13}{5}\vec{g}_1 = \frac{1}{5}\begin{bmatrix} 39 \\ 52 \\ 0 \end{bmatrix}$$

3. What is the projection of $\vec{b}$ onto $\vec{g}_2$

$$\text{proj}_{\vec{g}_2}(\vec{b}) = \frac{\vec{b} \cdot \vec{g}_2}{\vec{g}_2 \cdot \vec{g}_2}\vec{g}_2 = \frac{\frac{-28}{5} + \frac{33}{5}}{\frac{16}{25} + \frac{9}{25}}\vec{g}_2 = \frac{1}{1}\vec{g}_2 = 1\,\vec{g}_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

4. Define $\vec{g}_3 = \vec{b} - \text{proj}_{\vec{g}_1}(\vec{b}) - \text{proj}_{\vec{g}_2}(\vec{b})$

$$\vec{g}_3 = \frac{1}{5}\begin{bmatrix} 35 \\ 55 \\ 65 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 39 \\ 52 \\ 0 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 65 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix}$$

5. If you write $\vec{b} = y_1\vec{g}_1 + y_2\vec{g}_2 + y_3\vec{g}_3$, what are y_1, y_2, and y_3?

$$\vec{b} = \frac{13}{5}\vec{g}_1 + 1\,\vec{g}_2 + \vec{g}_3 \qquad y_1 = \frac{13}{5},\ y_2 = 1,\ y_3 = 1$$

6. Write each $\vec{v}_i$ in terms of the $\vec{g}_j$s

$$\vec{v}_1 = \vec{g}_1, \quad \vec{v}_2 = \vec{g}_2 + \frac{3}{5}\vec{g}_1$$

7. Find the best x_1, x_2 so that $x_1\vec{v}_1 + x_2\vec{v}_2$ is as close to $\vec{b}$ as possible.

$$x_1\vec{v}_1 + x_2\vec{v}_2 = x_1\vec{g}_1 + x_2\left(\vec{g}_2 + \frac{3}{5}\vec{g}_1\right) = \left(x_1 + \frac{3}{5}x_2\right)\vec{g}_1 + x_2\vec{g}_2$$

$$x_1 + \frac{3}{5}x_2 = \frac{13}{5}, \quad \boxed{x_2 = 1, \quad x_1 = 2}$$

8. How far from $\vec{b}$ must it be?

$$\|A\vec{x} - \vec{b}\| = \left\| \begin{bmatrix} 7 \\ 11 \\ 0 \end{bmatrix} - \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix} \right\| = \sqrt{0^2 + 0^2 + 13^2}$$

$$= \boxed{13}$$