MA322-001 Apr 11 Cliff Notes - Properties of orthogonal projection

Let $\vec{\mathbf{g}}_1$, $\vec{\mathbf{g}}_2$, $\vec{\mathbf{g}}_3$ be vectors in a vector space with $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ if $i \neq j$ and with $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i \neq 0$.
Set $\vec{\mathbf{v}} = x_1\vec{\mathbf{g}}_1 + x_2\vec{\mathbf{g}}_2 + x_3\vec{\mathbf{g}}_3$ and find an expression for $\|\vec{\mathbf{v}}\|^2 = \|x_1\vec{\mathbf{g}}_1 + x_2\vec{\mathbf{g}}_2 + x_3\vec{\mathbf{g}}_3\|^2$

Find an expression for $\vec{\mathbf{v}} \cdot \vec{\mathbf{g}}_i$

Do you need $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ for $i \neq j$?

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ be vectors in a vector space where $\langle \vec{\mathbf{v}}, \vec{\mathbf{v}} \rangle \neq 0$ and $\langle \vec{\mathbf{a}}, \vec{\mathbf{b}} \rangle = \langle \vec{\mathbf{b}}, \vec{\mathbf{a}} \rangle$.

Define $\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}} \vec{\mathbf{v}}$.

We have the following properties:

- $\vec{\mathbf{w}} \mapsto \mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}})$ is linear
- $\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \mathrm{proj}_{\vec{\mathbf{v}}}(\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}))$
- If $\vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{a}} \cdot \mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = 0$
- If $\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} = 0$, then $\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) \cdot \vec{\mathbf{a}} = 0$
- $\vec{\mathbf{v}} \cdot \mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$
- $\vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} - \mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{w}})) = 0$

Gram-Schmidt: Replace a sequence of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots$ with a sequence of orthogonal vectors $\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \ldots$ with the same span.

For $i = 1, 2, \ldots$
    Set $\vec{\mathbf{g}}_i = \vec{\mathbf{v}}_i$
    For $j = 1, 2, \ldots, i-1$
        Set $r_{ij} = \frac{\vec{\mathbf{g}}_j \cdot \vec{\mathbf{g}}_i}{r_{jj}}$ if $r_{jj} \neq 0$
        Set $\vec{\mathbf{g}}_i = \vec{\mathbf{g}}_i - r_{ij}\vec{\mathbf{g}}_j$
    End For
    Set $r_{ii} = \vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i$
End For
For $i = 1, 2, \ldots$
    Set $r_{ii} = 1$
End For

At the end of the ith iteration, $\vec{\mathbf{v}}_i = r_{i1}\vec{\mathbf{g}}_1 + \ldots + r_{(i-1),1}\vec{\mathbf{g}}_{i-1} + \vec{\mathbf{g}}_i$ and $\vec{\mathbf{g}}_j \cdot \vec{\mathbf{g}}_k = 0$ for all $1 \leq j < k \leq i$. If $r_{ii} = 0$, then $\vec{\mathbf{v}}_i$ is linearly dependent on the previous $\vec{\mathbf{v}}_j$, so $\vec{\mathbf{g}}_i = \vec{\mathbf{0}}$ could be omitted (or replaced with a random vector orthogonal to the previous vectors).

MA322-001 Apr 11 Example - Gram-Schmidt and Least Squares

Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 16 \\ 7 \\ 10 \end{bmatrix}$, and let $A = \begin{bmatrix} \uparrow & \uparrow \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \\ \downarrow & \downarrow \end{bmatrix}$. Set $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$ and $\vec{\mathbf{x}}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

We want to find x_1 and x_2 so that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 = \vec{\mathbf{b}}$, that is, find $\vec{\mathbf{x}}$ so that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

We cannot directly use dot products, because the columns of A are not orthogonal. We will replace A with a matrix G that does have orthogonal columns.

We'll keep the first column $\vec{\mathbf{g}}_1 = \vec{\mathbf{v}}_1$, but the second column of A points in the $\vec{\mathbf{g}}_1$ direction:

$\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{g}}_1 = 162$ $\quad$ $\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_1 = 81$ $\quad$ $\mathrm{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{v}}_2) = \frac{162}{81}\vec{\mathbf{g}}_1 = 2\vec{\mathbf{g}}_1 = \begin{bmatrix} 8 \\ 8 \\ 14 \end{bmatrix}$.

If we set $\vec{\mathbf{g}}_2 = \vec{\mathbf{v}}_2 - \mathrm{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{v}}_2) = \begin{bmatrix} 8 \\ -1 \\ -4 \end{bmatrix}$ then

$\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_2 = 0$ $\quad$ $\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_2 = 81$ $\quad$ $\vec{\mathbf{v}}_2 = 2\vec{\mathbf{g}}_1 + \vec{\mathbf{g}}_2$

How does this relate to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Well we can write $A\vec{\mathbf{x}}$ in terms of G:

$$A\vec{\mathbf{x}} = x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 = x_1\vec{\mathbf{g}}_1 + x_2(2\vec{\mathbf{g}}_1 + \vec{\mathbf{g}}_2) = (x_1 + 2x_2)\vec{\mathbf{g}}_1 + (x_2)\vec{\mathbf{g}}_2$$

And we can almost write $\vec{\mathbf{b}}$ in terms of G:

$\vec{\mathbf{b}} \cdot \vec{\mathbf{g}}_1 = 63$ $\quad$ $\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_1 = 81$ $\quad$ $\mathrm{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{b}}) = \frac{63}{81}\vec{\mathbf{g}}_1 = \frac{7}{9}\vec{\mathbf{g}}_1 = \frac{1}{9}\begin{bmatrix} 28 \\ 28 \\ 49 \end{bmatrix}$.

$\vec{\mathbf{b}} \cdot \vec{\mathbf{g}}_2 = -18$ $\quad$ $\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_2 = 81$ $\quad$ $\mathrm{proj}_{\vec{\mathbf{g}}_2}(\vec{\mathbf{b}}) = \frac{-18}{81}\vec{\mathbf{g}}_1 = \frac{-2}{9}\vec{\mathbf{g}}_2 = \frac{1}{9}\begin{bmatrix} -16 \\ 2 \\ 8 \end{bmatrix}$.

Now we get sneaky, and set $\vec{\mathbf{g}}_3 = \vec{\mathbf{b}} - \mathrm{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{b}}) - \mathrm{proj}_{\vec{\mathbf{g}}_2}(\vec{\mathbf{b}}) = \frac{1}{3}\begin{bmatrix} -1 \\ 8 \\ -4 \end{bmatrix}$

$\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_3 = 0$ $\quad$ $\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_3 = 0$ $\quad$ $\vec{\mathbf{g}}_3 \cdot \vec{\mathbf{g}}_3 = 9$ $\quad$ $\vec{\mathbf{b}} = \frac{7}{9}\vec{\mathbf{g}}_1 + \frac{-2}{9}\vec{\mathbf{g}}_2 + \vec{\mathbf{g}}_3$

Calculate $\|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|$ using the $\vec{\mathbf{g}}$s:

$$\begin{aligned} \|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|^2 &= \| \left((x_1 + 2x_2)\vec{\mathbf{g}}_1 + (x_2)\vec{\mathbf{g}}_2\right) - \left(\tfrac{7}{9}\vec{\mathbf{g}}_1 + \tfrac{-2}{9}\vec{\mathbf{g}}_2 + \vec{\mathbf{g}}_3\right) \|^2 \\ &= \| \left(x_1 + 2x_2 - \tfrac{7}{9}\right)\vec{\mathbf{g}}_1 + \left(x_2 - \tfrac{-2}{9}\right)\vec{\mathbf{g}}_2 + \vec{\mathbf{g}}_3\|^2 \\ &= (x_1 + 2x_2 - \tfrac{7}{9})^2\|\vec{\mathbf{g}}_1\|^2 + (x_2 - \tfrac{-2}{9})^2\|\vec{\mathbf{g}}_2\|^2 + \|\vec{\mathbf{g}}_3\|^2 \end{aligned}$$

This is clearly minimized exactly when $x_1 + 2x_2 = \frac{7}{9}$ and $x_2 = \frac{-2}{9}$, that is, when $\vec{\mathbf{x}} = \frac{1}{9}\begin{bmatrix} 11 \\ -2 \end{bmatrix}$, but it can never be smaller than $\|\vec{\mathbf{g}}_3\|^2 = 9$.

MA322-001 Apr 11 Quiz Name: ______________________

Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, and $\vec{\mathbf{b}} = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}$.

1. Find vectors $\vec{\mathbf{g}}_1$ and $\vec{\mathbf{g}}_2$ with the same span as $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, except with $\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_2 = 0$.

2. What is the projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{g}}_1$

3. What is the projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{g}}_2$

4. Define $\vec{\mathbf{g}}_3 = \vec{\mathbf{b}} - \mathrm{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{b}}) - \mathrm{proj}_{\vec{\mathbf{g}}_2}(\vec{\mathbf{b}})$

5. If you write $\vec{\mathbf{b}} = y_1\vec{\mathbf{g}}_1 + y_2\vec{\mathbf{g}}_2 + y_3\vec{\mathbf{g}}_3$, what are y_1, y_2, and y_3?

6. Write each $\vec{\mathbf{v}}_i$ in terms of the $\vec{\mathbf{g}}_j$s

7. Find the best x_1, x_2 so that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2$ is as close to $\vec{\mathbf{b}}$ as possible.

8. How far from $\vec{\mathbf{b}}$ must it be?