Real

MA322-001 Apr 14 Practice Exam Name: Key

1. Inner products and orthogonality - Show work clearly.

(a) Compute the dot product of $\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{a}}_2 = \begin{bmatrix} 100 \\ 10 \\ -1 \end{bmatrix}$

$1(100) + 2(10) + 3(-1) = 100 + 20 - 3 = 117$

(b) Suppose $\vec{\mathbf{b}}_1 \cdot \vec{\mathbf{b}}_1 = 1000000$, $\vec{\mathbf{b}}_1 \cdot \vec{\mathbf{b}}_2 = \vec{\mathbf{b}}_2 \cdot \vec{\mathbf{b}}_1 = 1000$, and $\vec{\mathbf{b}}_2 \cdot \vec{\mathbf{b}}_2 = 1$. Compute $(1\vec{\mathbf{b}}_1 + 3\vec{\mathbf{b}}_2) \cdot (5\vec{\mathbf{b}}_1 + 7\vec{\mathbf{b}}_2)$.

$(1)(5)\, b_1 \cdot b_1 + \big((1)(7) + (3)(5)\big)\, b_1 \cdot b_2 + (3)(7)\, b_2 \cdot b_2$

$= 5\,000000 + (7 + 15)\,1000 + 21$

$= 5022021$

(c) Suppose $\vec{\mathbf{c}}_1 \cdot \vec{\mathbf{c}}_1 = 1$, $\vec{\mathbf{c}}_1 \cdot \vec{\mathbf{c}}_2 = 0$, and $\vec{\mathbf{c}}_2 \cdot \vec{\mathbf{c}}_2 = 1$. Compute $(1\vec{\mathbf{c}}_1 + 3\vec{\mathbf{c}}_2) \cdot (5\vec{\mathbf{c}}_1 + 7\vec{\mathbf{c}}_2)$.

$(1)(5)\, c_1 \cdot c_1 + (\quad)\, 0 + (3)\cancel{(7)\, c_2 \cdot c_2}$

$= 5 + 21 = 26$

(d) Suppose $\|\vec{\mathbf{d}}_1\| = 1$, $\vec{\mathbf{d}}_1 \cdot \vec{\mathbf{d}}_2 = 0$, and $\|\vec{\mathbf{d}}_2\| = 1$. Compute $\|8\vec{\mathbf{d}}_1 + 15\vec{\mathbf{d}}_2\|$.

$\|8\vec{d}_1 + 15\vec{d}_2\|^2 = (8\vec{d}_1 + 15\vec{d}_2) \cdot (8\vec{d}_1 + 15\vec{d}_2)$

$= 8^2 + 15^2 = 64 + 225 = 289 = 17^2$

$\|8\vec{d}_1 + 15\vec{d}_2\| = 17$

(e) Give an example of two vectors $\vec{\mathbf{f}}_1$ and $\vec{\mathbf{f}}_2$ with no 0s in their coordinates with $\vec{\mathbf{f}}_1 \cdot \vec{\mathbf{f}}_2 = 0$.

$\begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ \pi \end{bmatrix}$

2. Projections - Show work clearly.

(a) Suppose $\vec{\mathbf{a}}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{\mathbf{a}}_2 = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. Compute the projection of $\vec{\mathbf{a}}_2$ onto $\vec{\mathbf{a}}_1$.

$$\text{proj}_{\vec{a}_1}(\vec{a}_2) = \frac{\vec{a}_1 \cdot \vec{a}_2}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 = \frac{6+44}{9+16} \vec{a}_1 = \frac{50}{25} \vec{a}_1 = 2\vec{a}_1 = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

(b) Suppose $\vec{\mathbf{b}}_1 \cdot \vec{\mathbf{b}}_1 = 2$, $\vec{\mathbf{b}}_1 \cdot \vec{\mathbf{b}}_2 = 14$, and $\vec{\mathbf{b}}_2 \cdot \vec{\mathbf{b}}_2 = 23$. Compute the projection of $\vec{\mathbf{b}}_2$ onto $\vec{\mathbf{b}}_1$.

$$\text{proj}_{\vec{b}_1}(\vec{b}_2) = \frac{\vec{b}_1 \cdot \vec{b}_2}{\vec{b}_1 \cdot \vec{b}_1} \vec{b}_1 = \frac{14}{2} \vec{b}_1 = 7\vec{b}_1$$

(c) Suppose $\vec{\mathbf{c}}_1 \cdot \vec{\mathbf{c}}_1 = 2$, $\vec{\mathbf{c}}_1 \cdot \vec{\mathbf{c}}_2 = 0$, and $\vec{\mathbf{c}}_2 \cdot \vec{\mathbf{c}}_2 = 23$. Compute the projection of $\vec{\mathbf{c}}_2$ onto $\vec{\mathbf{c}}_1$.

$$\text{proj}_{\vec{c}_1}(\vec{c}_2) = \frac{\vec{c}_1 \cdot \vec{c}_2}{\vec{c}_1 \cdot \vec{c}_1} \vec{c}_1 = \frac{0}{2} \vec{c}_1 = \vec{0}$$

(d) Let $\vec{\mathbf{d}} = \begin{bmatrix} 2 \\ 14 \\ 23 \end{bmatrix}$. Compute the projection of $\vec{\mathbf{d}}$ onto $\vec{\mathbf{d}}$.

$$\text{proj}_{\vec{d}}(\vec{d}) = \frac{\vec{d} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \cancel{?}\,\vec{d} = \boxed{\vec{d}}$$

(e) Suppose $\vec{\mathbf{f}}_1 = \begin{bmatrix} 2 \\ 14 \\ 23 \end{bmatrix}$ and $\vec{\mathbf{f}}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Compute the projection of $\vec{\mathbf{f}}_2$ onto $\vec{\mathbf{f}}_1$.

$$\text{proj}_{\vec{f}_1}(\vec{f}_2) = \frac{0+0+0}{?} \vec{f}_1 = \vec{0}$$

3. Gram-Schmidt - Show work clearly.

(a) Let $\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{\mathbf{a}}_2 = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\mathbf{a}}_3 = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$. Compute vectors $\vec{\mathbf{g}}_1$, $\vec{\mathbf{g}}_2$, and $\vec{\mathbf{g}}_3$ that are orthogonal and span the same subspace as $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, and $\vec{\mathbf{a}}_3$ using the Gram-Schmidt procedure.

$\vec{g}_1 = \vec{a}_1$ no change (can use $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$ too, but just makes it messier)

$$\vec{g}_2 = \vec{a}_2 - \text{proj}_{\vec{g}_1}(\vec{a}_2)$$

$$= \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1} \vec{g}_1$$

$$= \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{5+3+0+0}{1+1+0+0} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} - \frac{8}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{g}_3 = \vec{a}_3 - \text{proj}_{\vec{g}_1}(\vec{a}_3) - \text{proj}_{\vec{g}_2}(\vec{a}_3)$$

$$= \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix} - \frac{9+9+0+0}{1+1+0+0} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{9-9+9+0}{1+1+1+0} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 3 \\ 6 \\ 9 \end{bmatrix}$$

4. Least Squares - Show work clearly.

(a) Suppose $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{a}_3 = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$. Notice that $\vec{a}_1 \cdot \vec{a}_1 = 10$, $\vec{a}_1 \cdot \vec{a}_2 = 0$, $\vec{a}_1 \cdot \vec{a}_3 = 20$, $\vec{a}_2 \cdot \vec{a}_2 = 1$, and $\vec{a}_2 \cdot \vec{a}_3 = 6$. Find $y_1$ and $y_2$ so that $y_1\vec{a}_1 + y_2\vec{a}_2$ is as close as possible to $\vec{a}_3$.

Use orthogaral projection

$$\vec{a}_3 = \text{proj}_{\vec{a}_1}(\vec{a}_3) + \text{proj}_{\vec{a}_2}(\vec{a}_3) + (\text{err orth to } \vec{a}_1 \text{ and } \vec{a}_2)$$

$$= \frac{a_1 \cdot a_3}{a_1 \cdot a_1} a_1 + \frac{a_2 \cdot a_3}{a_2 \cdot a_2} a_2 + \text{—}$$

$$y_1 = \frac{a_1 \cdot a_3}{a_1 \cdot a_1} = \frac{20}{10} = 2$$

$$y_2 = \frac{6}{1} = 6$$

(b) How close can $y_1\vec{a}_1 + y_2\vec{a}_2$ be to $\vec{a}_3$?

Method 1

$$\sqrt{(8-y_1)^2 + (6-y_2)^2 + (4-3y_1)^2}$$

$$= \sqrt{64 - 16y_1 + y_1^2 + 16 - 24y_1 + 9y_1^2 + (6-y_2)^2}$$

$$= \sqrt{16(y_1-2)^2 + (6-y_2)^2 + 40}$$

Method 2

$$\text{err} = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - 6\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$$

$$\|\text{err}\| = \sqrt{6^2 + 2^2} = \sqrt{40}$$

is min at $y_1 = 2$, $y_2 = 6$, $\sqrt{40}$

(c) $\vec{c}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{c}_3 = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$. How close can $x_1\vec{c}_1 + x_2\vec{c}_2$ be to $\vec{c}_3$?

Choose $x_1, x_2$ so that $x_1\vec{c}_1 + x_2\vec{c}_2 = \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix}$

err is $\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$, $\|\text{err}\| = 4$ (Done)

Bonus: $\vec{c}_3 = \frac{8+6+0}{1+1+0}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{8-6+0}{1+1+0}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \text{err}$

$$= \frac{14}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= 7\vec{c}_1 + 1(\vec{c}_2 - 2\vec{c}_1)$$

$$= 5\vec{c}_1 + 1\vec{c}_2$$

Bonus: What $x_1$ and $x_2$ work to get that closest value? Hint: $\vec{c}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2\vec{c}_1$.

$x_1 = 5$, $x_2 = 1$