MA322-001 Apr 18 Quiz and cliff notes Name: Key

A **rank one** matrix is a matrix of the form uv^T for column vectors u and v. This is the opposite order of the dot product, $u^Tv = \langle u, v\rangle$. If u has m rows, and v has n rows then uv^T is a $m \times n$ matrix with m rows and n columns.

Today we only care about the case when $u = v$, where something magical happens!

Let $u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Find the matrix $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$. $= \begin{bmatrix} 1\cdot1 & 1\cdot0 & 1\cdot2 \\ 0\cdot1 & 0\cdot0 & 0\cdot2 \\ 2\cdot1 & 2\cdot0 & 2\cdot2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

What are its eigenpairs? Use Ch5

$\left(5, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) \quad \left(0, \begin{bmatrix} -2z \\ y \\ z \end{bmatrix}\right)$ for any y, z not both zero

What do you notice about the matrix?

Symmetric $(\vec{u}\vec{u}^T)^T = \vec{u}\vec{u}^T$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

What do you notice about its eigenvectors?

$\vec{u}\vec{u}^T$ has eigenpairs $(\vec{u}^T\vec{u}, \vec{u})$ and $(0, \vec{v})$ for any nonzero $\vec{v}$ orthogonal to $\vec{u}$

Symmetric matrices

A matrix A is **symmetric** iff $A = A^T$. Symmetric matrices have to be square, so they can have eigenvalues and eigenvectors. In fact, the eigenvalues and eigenvectors are very nice: The eigenvalues are all real numbers (no square roots of negatives) and the eigenvectors for distinct eigenvalues are orthogonal (so easy to project onto). If A is $n \times n$, then it has exactly n eigenvalues (counted with multiplicity) and $\mathbb{R}^n$ has an orthogonal basis of eigenvectors.

Symmetric matrices are exactly the $n \times n$ matrices that can be written as a sum

$$c_1\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T + c_2\vec{\mathbf{u}}_2\vec{\mathbf{u}}_2^T + c_3\vec{\mathbf{u}}_3\vec{\mathbf{u}}_3^T + \ldots c_n\vec{\mathbf{u}}_n\vec{\mathbf{u}}_n^T$$

with c_i real numbers (the eigenvalues) and $\vec{\mathbf{u}}_i$ unit-length **orthogonal** column vectors with n rows (the eigenvectors). If you use less than n vectors, the remaining ones can be taken to be $\vec{\mathbf{0}}$. If $\vec{\mathbf{u}}_i$ is not unit-length, then the eigenvalue is actually $c_i\vec{\mathbf{u}}_i^T\vec{\mathbf{u}}_i$, the c_i multiplied by the length squared.