

A **rank one** matrix is a matrix of the form uv^T for column vectors u and v . This is the opposite order of the dot product, $u^T v = \langle u, v \rangle$. If u has m rows, and v has n rows then uv^T is a $m \times n$ matrix with m rows and n columns.

Today we only care about the case when $u = v$, where something magical happens!

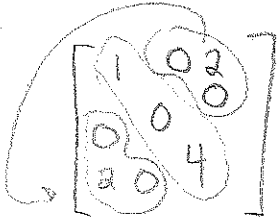
Let $u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Find the matrix $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot [1 \ 0 \ 2] = \begin{bmatrix} 1 \cdot 1 & 1 \cdot 0 & 1 \cdot 2 \\ 0 \cdot 1 & 0 \cdot 0 & 0 \cdot 2 \\ 2 \cdot 1 & 2 \cdot 0 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

What are its eigenpairs? Use Ch5

$\left(5, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) \quad \left(0, \begin{bmatrix} -2z \\ y \\ z \end{bmatrix}\right)$ for any y, z not both zero

What do you notice about the matrix?

Symmetric $(\vec{u}\vec{u}^T)^T = \vec{u}\vec{u}^T$



What do you notice about its eigenvectors?

$\vec{u}\vec{u}^T$ has eigenpairs $(\vec{u}^T\vec{u}, \vec{u})$ and $(0, \vec{v})$
for any nonzero \vec{v} orthogonal to \vec{u}

Symmetric matrices

A matrix A is **symmetric** iff $A = A^T$. Symmetric matrices have to be square, so they can have eigenvalues and eigenvectors. In fact, the eigenvalues and eigenvectors are very nice: The eigenvalues are all real numbers (no square roots of negatives) and the eigenvectors for distinct eigenvalues are orthogonal (so easy to project onto). If A is $n \times n$, then it has exactly n eigenvalues (counted with multiplicity) and \mathbb{R}^n has an orthogonal basis of eigenvectors.

Symmetric matrices are exactly the $n \times n$ matrices that can be written as a sum

$$c_1 \vec{u}_1 \vec{u}_1^T + c_2 \vec{u}_2 \vec{u}_2^T + c_3 \vec{u}_3 \vec{u}_3^T + \dots c_n \vec{u}_n \vec{u}_n^T$$

with c_i real numbers (the eigenvalues) and \vec{u}_i unit-length **orthogonal** column vectors with n rows (the eigenvectors). If you use less than n vectors, the remaining ones can be taken to be $\vec{0}$. If \vec{u}_i is not unit-length, then the eigenvalue is actually $c_i \vec{u}_i^T \vec{u}_i$, the c_i multiplied by the length squared.