**Today's topic:** A quadratic form takes  $\vec{\mathbf{x}}$  to  $\langle \vec{\mathbf{x}}, A\vec{\mathbf{x}} \rangle = \vec{\mathbf{x}} \cdot A\vec{\mathbf{x}}$ . Compare this to a linear transformation that takes  $\vec{\mathbf{x}}$  to  $A\vec{\mathbf{x}}$ . Informally, linear transformations tell you coordinates, while quadratic forms tell you energy.

**Important example:** The sample mean  $M_n$  and the sample variance  $V_n$  are important statistics defined on a series of observations  $\vec{\mathbf{x}}$ . The sample mean is linear:

$$M_n = \frac{1}{n} \sum_{i=1}^n x_i$$
 (the average)

The sample variance is a quadratic form:

$$V_n = \frac{1}{n-1} \sum_{i=1}^n (x_i - M_n)^2$$

For n = 2, we get

$$V_2(x,y) = \frac{1}{1} \left( (x - (x+y)/2)^2 + (y - (x+y)/2)^2 \right) = \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$$

This is the same as

$$V_2 = \vec{\mathbf{x}} \cdot A\vec{\mathbf{x}}$$
 for  $\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$ 

Is there a simpler formula for  $V_2$ ? Yes, there is and we'll use chapter 5-7 to find it! The formula  $\frac{1}{2}x^2 - xy + \frac{1}{2}y^2$  is ugly because "x" and "y" are the wrong variables. We need to choose better variables; we need to choose eigenvectors.

Find the eigenpairs of A. I'll call them  $(c_0, \vec{\mathbf{v}}_0)$  and  $(c_1, \vec{\mathbf{v}}_1)$ .

Every  $\vec{\mathbf{x}} = (x, y)$  can be written as the sum of two vectors:  $\frac{\vec{\mathbf{v}}_0 \cdot \vec{\mathbf{x}}}{\vec{\mathbf{v}}_0 \cdot \vec{\mathbf{v}}_0} \vec{\mathbf{v}}_0$  added to  $\frac{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{x}}}{\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1} \vec{\mathbf{v}}_1$ . Simplify those two vectors using your values for  $\vec{\mathbf{v}}_i$ .

 $V_2 = \vec{\mathbf{x}}^T A \vec{\mathbf{x}} = (a \vec{\mathbf{v}}_0^T + b \vec{\mathbf{v}}_1^T) (A(a \vec{\mathbf{v}}_0 + b \vec{\mathbf{v}}_1))$ That means that  $= (a \vec{\mathbf{v}}_0^T + b \vec{\mathbf{v}}_1^T) (a c_0 \vec{\mathbf{v}}_0 + b c_1 \vec{\mathbf{v}}_1)$ Simplify this using your values for  $a, b, c_i$ , and  $\vec{\mathbf{v}}_i$ . $= a^2 c_0 \|\vec{\mathbf{v}}_0\|^2 + b^2 c_1 \|\vec{\mathbf{v}}_1\|^2.$  Main Idea of 7.2: The principal axes of a quadratic form are the eigenvectors of its associated symmetric matrix. If  $\vec{\mathbf{x}} \cdot A\vec{\mathbf{x}}$  is the quadratic form,  $(c_i, \vec{\mathbf{v}}_i)$  are its eigenpairs, then the spectral decomposition of A is

$$A = \sum_{i=1}^{n} \frac{c_i}{\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i} \vec{\mathbf{v}}_i^T \quad \text{and} \quad \vec{\mathbf{x}} \cdot A\vec{\mathbf{x}} = \sum_{i=1}^{n} \frac{c_i}{\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i} (\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i)^2$$

**Important example:** Let's examine sample variance for n samples, instead of just 2.

$$V_n = \vec{\mathbf{x}} \cdot A\vec{\mathbf{x}}$$
 for  $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  and  $A = \frac{1}{n-1}I - \frac{1}{n(n-1)}J$ 

where I is the identity matrix and J is the matrix of all ones (a very famous matrix).

We can find an orthogonal basis of eigenvectors using chapter 5: The first eigenpair is  $(c_1, \vec{\mathbf{v}}_1) = (0, \vec{\mathbf{j}})$  where  $\vec{\mathbf{j}}$  is the vector of all ones. The other eigenvalues are all  $\frac{1}{n-1}$ , so we have to be a little clever to find orthogonal eigenvectors. I'll use the output of Gram-Schmidt when applied to the consuecutive difference eigenvectors  $\vec{\mathbf{e}}_{i-1} - \vec{\mathbf{e}}_i$ . We get the *i*th eigenpair is  $(c_i, \vec{\mathbf{v}}_i)$  where  $c_i = \frac{1}{n-1}$  and  $\vec{\mathbf{v}}_i = \sum_{j=1}^{i-1} (e_j - e_i)$  is the vector that is all ones up to the (i-1)st coordinate, is 1-i as the *i*th coordinate, and zero afterwards.

Here are the relevant dot products (here i = 2, 3, ..., n):

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_1 = x_1 + x_2 + \dots + x_n = nM_n \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1 = 1^2 + 1^2 + \dots + 1^2 = n \vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i = x_1 + x_2 + \dots + x_{i-1} - (i-1)x_i = (i-1)(M_{i-1} - x_i) \vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i = 1^2 + 1^2 + \dots + 1^2 + (i-1)^2 = (i-1) + (i-1)^2 = i(i-1)$$

Using the principal axes of A we get

$$V_n = \vec{\mathbf{x}} \cdot A\vec{\mathbf{x}} = \sum_{i=1}^n \frac{c_i}{\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i} (\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i)^2 = \frac{1}{n-1} \sum_{i=2}^n (M_{i-1} - x_i)^2 (1 - 1/i)$$

Notice that

$$V_{n+1} = (1 - 1/n)V_n + (M_n - x_{n+1})^2/(n+1)$$

This gives a simple, numerically stable, memory-less algorithm, published as Welford (1962), to update a running variance:

```
function [ new_count, new_ave, new_var] = init( new_obs )
    new_count = 1
    new_ave = new_obs
    new_var = 0
function [ new_count, new_ave, new_var] = update( old_count, old_ave, old_var, new_obs )
    new_count = old_count + 1;
    new_ave = old_ave + (new_obs - old_ave)/new_count
    new_var = old_var*(1-1/old_count) + ( old_ave - new_obs )^2/(new_count);
```

This demonstrates how eigenvectors can be unexpectedly useful (especially in real time computing such as aircraft control).

MA322-001 Apr 21 Quiz

Name: \_\_\_\_\_

7.1: Let *A* be the diagonal matrix  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ .

(a) Compute the eigenpairs  $(c_i, \vec{\mathbf{v}}_i)$  of A:

(b) Compute the rank one matrices  $\frac{c_i}{\vec{\mathbf{v}}_i\cdot\vec{\mathbf{v}}_i}\vec{\mathbf{v}}_i\vec{\mathbf{v}}_i^T$ 

(c) Add them up to get the spectral decomposition of A:

7.2: Find a matrix B so that if 
$$\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$$
 then  $\vec{\mathbf{x}} \cdot B\vec{\mathbf{x}} = x^2 + 14xy + 5y^2$ .

(b) Actually multiply out  $B\vec{\mathbf{x}}$ 

(c) Actually multiply out  $\vec{\mathbf{x}}\cdot B\vec{\mathbf{x}}$