Today's topic: Maximizing and minimizing quadratic forms.

The spectral decomposition of a symmetric matrix (the principal axes of the quadratic form) make evaluating the quadratic form easy in terms of the orthogonal eigenvector basis: If $\vec{x} \cdot A\vec{x}$ is the quadratic form, $(c_i, \vec{v}_i)$ are its eigenpairs, then the spectral decomposition of A is

$$A = \sum_{i=1}^{n} \frac{c_i}{\vec{v}_i \cdot \vec{v}_i} \vec{v}_i \vec{v}_i^T \quad \text{and} \quad \vec{x} \cdot A\vec{x} = \sum_{i=1}^{n} \frac{c_i}{\vec{v}_i \cdot \vec{v}_i} (\vec{x} \cdot \vec{v}_i)^2$$

There are very mature and effective algorithms to find eigenvalues of symmetric matrices. It takes about 1 second to find all 1500 eigenvalues of a 1500×1500 dense symmetric matrix. It takes about 1 second to find all 500 eigenpairs of a 500×500 dense symmetric matrix.

In chapter 6, we used projection to pick out parts of the answer, and find the length of the error. Now we will use the same idea to find the most important part of a vector for a quadratic form.

The larger a vector, the larger the quadratic form becomes.

(a) Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$. Calculate $Q(7\vec{x})$ in terms of $Q(\vec{x})$:

$$Q(7\vec{x}) = (7\vec{x}^T)(A(7\vec{x})) = 7 \cdot 7 \cdot \vec{x}^T A\vec{x} = 7^2 Q(\vec{x})$$

For this reason, when finding the maximum value of a quadratic form, we must specify a maximum size for $\vec{x}$. Since $Q(x)$ can be calculated from c_i, and $\frac{(\vec{x} \cdot \vec{v}_i)^2}{\vec{v}_i \cdot \vec{v}_i}$ we want to see how these affect the length of $\vec{x}$.

(b) Assume $\vec{v}_i$ are orthogonal. Write $\vec{x}$ as a linear combination of the $\vec{v}_i$.

$$\vec{x} = \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 + \cdots$$

(c) Now compute the length (squared) of $\vec{x}$ in terms of $\frac{(\vec{x} \cdot \vec{v}_i)^2}{\vec{v}_i \cdot \vec{v}_i}$. $= t_i$

$$\|x\|^2 = \left(\frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)^2 \vec{v}_1 \cdot \vec{v}_1 + \left(\frac{\vec{x} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)^2 \vec{v}_2 \cdot \vec{v}_2 + \cdots = t_1 + t_2 + \cdots + t_n$$

(d) If we want $\vec{x}$ to have length 1, then what value of $t_i = \frac{(\vec{x} \cdot \vec{v}_i)^2}{\vec{v}_i \cdot \vec{v}_i}$ gives the largest value of $\sum c_i t_i$?

If c_1 is the largest of the c_i, then $t_1 = 0$
$t_2 = t_3 = \cdots = t_n = 0$

That means $\vec{x} = \frac{1}{\sqrt{\vec{v}_1 \cdot \vec{v}_1}} \vec{v}_1$

That is how to maximize a quadratic form.

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7.1: Let A be the diagonal matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

(a) Compute the eigenpairs $(c_i, \vec{v}_i)$ of A:

$$\left(1, \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}\right), \left(3, \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}\right), \left(5, \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}\right)$$

(b) Compute the rank one matrices $\frac{c_i}{\vec{v}_i \cdot \vec{v}_i} \vec{v}_i \vec{v}_i^T$

$$\frac{1}{x^2+0^2+0^2} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} [x\ 0\ 0] = \frac{1}{x^2} \begin{bmatrix} x \cdot x & x \cdot 0 & x \cdot 0 \\ 0 \cdot x & 0 \cdot 0 & 0 \cdot 0 \\ 0 \cdot x & 0 \cdot 0 & 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{3}{0^2+y^2+0} \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} [0\ y\ 0] = \frac{3}{y^2} \begin{bmatrix} 0 \cdot 0 & 0 \cdot y & 0 \cdot 0 \\ y \cdot 0 & y \cdot y & y \cdot 0 \\ 0 \cdot 0 & 0 \cdot y & 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Add them up to get the spectral decomposition of A:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{mul } x \text{ by } 1} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{mul } y \text{ by } 3} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{\text{mul } z \text{ by } 5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = A$$

7.2: Find a matrix B so that if $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $\vec{x}^T \cdot B\vec{x} = x^2 + 3y^2 + 5z^2$.

If $B = \begin{bmatrix} b_1 & b_{12} & b_{13} \\ b_{12} & b_2 & b_{23} \\ b_{13} & b_{23} & b_3 \end{bmatrix}$ then $\vec{x}^T B \vec{x} = b_1 x^2 + b_2 y^2 + b_3 z^2 + 2b_{12}xy + 2b_{13}xz + 2b_{23}yz$

(b) Actually multiply out $B\vec{x}$

$$Bx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x \\ 3y \\ 5z \end{bmatrix} \text{ so } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(c) Actually multiply out $\vec{x}^T \cdot B\vec{x}$

$$[x\ y\ z] \begin{bmatrix} 1x \\ 3y \\ 5z \end{bmatrix} = \left[1x^2 + 3y^2 + 5z^2\right]$$

7.3: Find the unique vector $\vec{x}$ with $||\vec{x}|| = 1$ such that both $\vec{x}^T \cdot A\vec{x}$ and $\vec{x}^T \cdot B\vec{x}$ are maximized.

The z has the best payoff. $z=1, x=0, y=0$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{x}^T A\vec{x} = 5$$