Today's topic: SVD - spectral decomposition for rectangular matrices

Chapter 6 showed us that orthogonal vectors turned all that row reduction into projection (dot products). Chapter 7.1-7.3 used this combined with chapter 5 to handle square symmetric matrices very nicely. Now we handle rectangular matrices (this course does not cover square non-symmetric matrices, like Markov matrices, sorry).

Main idea: For every matrix A there are two sets of orthogonal vectors $\{\vec{\mathbf{u}}_i\}$ and $\{\vec{\mathbf{v}}_i\}$ so that

$$A = \sum_{i=1}^{r} \vec{\mathbf{u}}_i \vec{\mathbf{v}}_i^T$$

Multiplication: To compute $A\vec{\mathbf{x}}$ we use dot-products: compute $\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i$ and use that as the $\vec{\mathbf{u}}_i$ coordinate:

$$A\vec{\mathbf{x}} = \left(\sum_{i=1}^{r} \vec{\mathbf{u}}_i \vec{\mathbf{v}}_i^T \right) \vec{\mathbf{x}} = \sum_{i=1}^{r} \vec{\mathbf{u}}_i \left(\vec{\mathbf{v}}_i^T \vec{\mathbf{x}} \right) = \sum_{i=1}^{r} (\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i) \vec{\mathbf{u}}_i$$

Lengths: Since everything is orthogonal, lengths are very easy to compute:

$$\|\vec{\mathbf{x}}\|^2 = \sum_{i=0}^{n} \frac{(\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_a)^2}{\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i} \qquad \|A\vec{\mathbf{x}}\|^2 = \sum_{i=0}^{r} (\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i)^2 \|u_i\|^2$$

If we set $t_i = \frac{(\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i)^2}{\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i}$ and $c_i = \|\vec{\mathbf{v}}_i\|^2 \|\vec{\mathbf{u}}_i\|^2$, then we get the following problem (example 1)

$$\text{Maximize } \|A\vec{\mathbf{x}}\| \text{ subject to } \|\vec{\mathbf{x}}\| \leq 1$$

which is the same as

$$\text{Maximize } \textstyle\sum_{i=0}^{r} c_i t_i \text{ subject to } \sum_{i=0}^{n} t_i \leq 1,\ t_i \geq 0$$

The solution is to choose the i with the largest c_i, and set $t_i = 1$ and $t_j = 0$ otherwise.

Least squares: To minimize $\|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|$ we want the ith coordinates to match, $\vec{\mathbf{x}} \cdot \vec{\mathbf{v}}_i = \frac{\vec{\mathbf{b}} \cdot \vec{\mathbf{u}}_i}{\vec{\mathbf{u}}_i \vec{\mathbf{u}}_i}$, in other words

$$\vec{\mathbf{x}} = \sum_{i=0}^{r} \frac{\vec{\mathbf{b}} \cdot \vec{\mathbf{u}}_i}{\|\vec{\mathbf{v}}_i\|^2 \|\vec{\mathbf{u}}_i\|^2} \vec{\mathbf{v}}_i$$

The number $\sqrt{c_i}$ is called the ith **singular value** and is often called an eigenvalue by non-mathematicians.

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7.4 Let A be the rectangular matrix $A = \begin{bmatrix} 6.0 & 8.0 \\ 12.4 & 15.7 \\ 23.6 & 32.3 \\ 35.6 & 48.3 \\ 48.4 & 63.7 \end{bmatrix}$ and let $\vec{\mathbf{b}} = \begin{bmatrix} 1.4 \\ 2.1 \\ 4.0 \\ 5.9 \\ 8.0 \end{bmatrix}$.

We want to solve $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(a) For $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ and $\vec{\mathbf{v}}_1 = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, compute $A_1 = \vec{\mathbf{u}}_1\vec{\mathbf{v}}_1^T$. Is it close to A?

(b) Compute $\vec{\mathbf{x}}_1 = \dfrac{\vec{\mathbf{b}}^T\vec{\mathbf{u}}_1}{\|\vec{\mathbf{u}}_1\|^2\|\vec{\mathbf{v}}_1\|^2}\vec{\mathbf{v}}_1$.

(c) Compute $A_1\vec{\mathbf{x}}_1$. Is it close to $\vec{\mathbf{b}}$?

(d) Use algebra to simplify the expression $A_1\vec{\mathbf{x}}_1$. Does it have a name in terms of $\vec{\mathbf{b}}$ and $\vec{\mathbf{u}}_1$?

$$A = \begin{bmatrix} 6.0 & 8.0 \\ 12.4 & 15.7 \\ 23.6 & 32.3 \\ 35.6 & 48.3 \\ 48.4 & 63.7 \end{bmatrix}, \vec{\mathbf{b}} = \begin{bmatrix} 1.4 \\ 2.1 \\ 4.0 \\ 5.9 \\ 8.0 \end{bmatrix}, A_1 = \begin{bmatrix} \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix}, x_1 = \begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix}.$$

(e) For $\vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0.4 \\ -0.3 \end{bmatrix}$, compute $A_2 = A_1 + \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2^T$. Is it close to A?

(f) Set $\vec{\mathbf{x}}_2 = \vec{\mathbf{x}}_1 + \dfrac{\vec{\mathbf{b}}^T \vec{\mathbf{u}}_2}{\|\vec{\mathbf{u}}_2\|^2 \|\vec{\mathbf{v}}_2\|^2} \vec{\mathbf{v}}_2$. Use algebra to simplify the expression $A_2 \vec{\mathbf{x}}_2$.

(g) Does $A_2 \vec{\mathbf{x}}_2$ have a name in terms of $\vec{\mathbf{b}}$ and $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$?

(h) Compute $A_2 \vec{\mathbf{x}}_2$. Is it close to $\vec{\mathbf{b}}$? Can we do any better with $A\vec{\mathbf{x}}$?