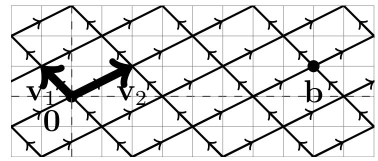
MA322-007 Jan 22 worksheet

Span

On Tuesday we covered several equivalent questions. For example solving these two systems was the same as tracing out a path on a grid:

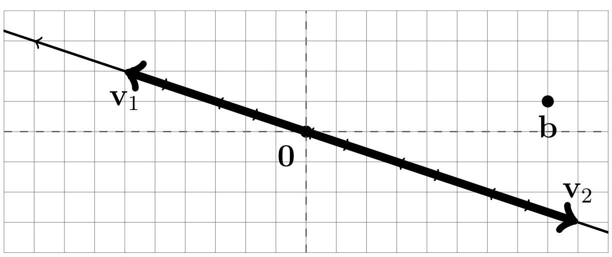
$$\begin{cases} -1x_1 + 2x_2 = 8 \\ 1x_1 + 1x_2 = 1 \end{cases} \qquad x_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$



Solving these two systems was the same as balancing this chemical reaction:

$$\begin{array}{l} 3x_1 - x_3 = 0 \\ 8x_1 - 2x_4 = 0 \\ 2x_2 - 2x_3 + x_4 = 0 \end{array} \qquad x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{array}{c} x_1\ C_3H_8 + x_2\ \ O_2 \\ \downarrow \\ x_3\ \ CO_2 + x_4\ H_2O \end{array}$$

However, some grid problems and some chemical reactions had no good solutions:



$$\underbrace{\qquad\quad} CH_4 + \underbrace{\qquad\quad} CO_2 \to \underbrace{\qquad\quad} H_2O$$

$$x_1 \begin{bmatrix} -6 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix} \qquad x_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

In all of these problems, we need to write a vector $\vec{\mathbf{b}}$ as a linear combination of some other vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, ... The **span** of a set of vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, ... is the collection of all linear combinations $\vec{\mathbf{b}} = x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + \ldots$ of those vectors.

The span of a single vector is usually a line. The span of two vectors is usually a plane.

Matrix equation

It can be convenient to gather all the vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, ... into a single matrix A, and all of the coefficients x_1, x_2, ... into a single vector $\vec{\mathbf{x}}$. Then the linear combination itself is written as $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$

$$A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \ldots & \vec{\mathbf{v}}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \qquad \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \ldots & \vec{\mathbf{v}}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + \ldots x_n\vec{\mathbf{v}}_n$$

Stoichiometry!

(a) Balance this reaction: $\underbrace{\quad\quad}C_3H_8 + \underbrace{\quad\quad}O_2 \to \underbrace{\quad\quad}CO_2 + \underbrace{\quad\quad}H_2O$

(b) Find scalars x_1, x_2, x_3, and x_4 (not all zero) so that

$$\underbrace{\quad x_1 \quad}\begin{bmatrix}3\\8\\0\end{bmatrix} + \underbrace{\quad x_2 \quad}\begin{bmatrix}0\\0\\2\end{bmatrix} = \underbrace{\quad x_3 \quad}\begin{bmatrix}1\\0\\2\end{bmatrix} + \underbrace{\quad x_4 \quad}\begin{bmatrix}0\\2\\1\end{bmatrix}$$

(c) Solve the equations

$$\begin{cases} 3x_1 = x_3 \\ 8x_2 = 2x_4 \\ 2x_2 = 2x_3 + x_4 \end{cases}$$

(d) Row reduce this matrix:

$$\left[\begin{array}{cccc|c} 3 & 0 & -1 & -0 & 0 \\ 8 & 0 & -0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

Hint: $\left[\begin{array}{cccc|c} 3 & 0 & -1 & -0 & 0 \\ 8 & 0 & -0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow[R_2-(8/3)R_1]{R_1/3, R_3/2} \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 8/3 & -2 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \end{array}\right] \xrightarrow[R_2\leftrightarrow R_3]{(3/8)R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{array}\right] \xrightarrow[R_2+R_3]{R_1+(1/3)R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{array}\right]$

(e) What is the general solution?

(f) What is a non-zero solution with all integers?

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HW1.3(#9) Write this system of equations as a single equation of vectors:
$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \\ -x_1 + 3x_2 + 8x_3 &= 0 \end{aligned}$$

HW1.3(#13) Write $\vec{\mathbf{b}}$ as a linear combination of the columns of A where
$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}$, $\vec{\mathbf{b}} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$.

1.4a Write HW1.3#9 as a single equation in the form $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

1.4b(HW1.3#22) Give an example of a matrix A and a vector $\vec{\mathbf{b}}$ so that the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution.