1.7: Linear Dependence MA322-007 Feb 3 Worksheet

Why does $x_1 \begin{bmatrix}1\\1\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\0\\1\\5\end{bmatrix} + x_3 \begin{bmatrix}2\\2\\1\\5\end{bmatrix} = \vec{b}$ either have no solutions or infinitely many solutions?

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

Since $\vec{v}_3 = 2\vec{v}_1 + \vec{v}_3$

Be prepared to answer to the class.

Any solution can be changed by adding $1\vec{v}_3$ and subtracting $2\vec{v}_1$ and $1\vec{v}_3$. Thus one solutions leads to infinitely many. But some $\vec{b}$ have no solutions at all.

"No" versus "infinitely many" depends on the numbers in $\vec{b}$ of course.
How do you tell which $\vec{b}$ have any solutions?
Be prepared to answer to the class.

If $\vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$ then there is one (and thus infinitely many) solution $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ if and only if

$b_1 = b_2$ (b_2 is free)

$b_4 = 5b_3$ (b_3 is free)

Takeaway: If the columns of a matrix A are linearly independent, then there is at most one solution $A\vec{x} = \vec{b}$, never infinitely many. The only way to get infinitely many is (the reason above; fill in the official version here).

Two solutions $\vec{x}$ and $\vec{y}$ mean $A\vec{x} = \vec{b}$ so subtract

$A\vec{y} = \vec{b}$

to get $A(\vec{x} - \vec{y}) = \vec{0}$

but $\vec{x} - \vec{y}$ is a linear dependence relation amongst the columns of A.

Weird question: You should have asked: "hrm, what about the rows? do the rows need to be linearly independent? What's up with that?" So... what IS up with that?

They don't need to be linearly independent, but any dependency relation of the rows of A must also hold for the rows of $\vec{b}$

1.7a) Example

If $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 15 \end{bmatrix}$ then $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is one solution

since $\vec{b} = 2\vec{v}_1 + 3\vec{v}_2$

but $\vec{b} = 2\vec{v}_1 + 3\vec{v}_2 + (\vec{v}_3 - 2\vec{v}_1 - \vec{v}_2)t$ for any t

so $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is also

a solution.

The "free" parts of the solution are the linear dependence relations between the <u>columns</u>

1.7b In general

The relations $\left\{\begin{matrix} b_1 = b_2 \\ b_4 = 5b_3 \end{matrix}\right\}$

are dependencies in the <u>rows</u> of $\vec{b}$.

They follow precisely from the dependencies in the rows of A.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{bmatrix} \qquad \begin{matrix} R_1 = R_2 \\ 5R_3 = R_4 \end{matrix}$$

How do we find these? Just do row reduction, but keep track of the "name" of the rows.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{bmatrix} \xrightarrow[R_4 - 5R_3]{R_2 - R_1} \begin{array}{c} R_1 \\ R_2 - R_1 \\ R_3 \\ R_4 - 5R_3 \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Zero Rows need to have zero $\vec{b}$ so the second row says $R_2 - R_1$ is zero, so $b_2 - b_1$ must be zero.

Row Reduction finds "all"* linear dependence relations of the row.

* (Technically it finds a linearly independent spanning set of linear dependence relations)

MA322-007 Feb 3 quiz Name: Key

1.7.1 Solve $A(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$ when $A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 5 & 4 \\ 3 & 5 & 1 \end{bmatrix}$, $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{\mathbf{b}} = \begin{bmatrix} 14 \\ 8 \\ 2 \end{bmatrix}$ using the observation that $-3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$.

You shouldn't need to row reduce.

I guess $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ works. Since $2\begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 2 \end{bmatrix}$

Then so does $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

because $-3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

1.7.2 In each matrix decide if the columns are linearly independent. If they are not, cross out the **minimum** number of columns to make them linearly independent.

$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$ LI

$\begin{bmatrix} 1 & 2 & \cancel{3} \\ 4 & 5 & \cancel{9} \\ 6 & 7 & \cancel{13} \end{bmatrix}$ LD $\vec{V}_3 = \vec{V}_1 + \vec{V}_2$

$\begin{bmatrix} \cancel{0} & 1 & \cancel{2} \\ \cancel{0} & 3 & \cancel{4} \\ \cancel{0} & 5 & \cancel{6} \end{bmatrix}$ $\vec{V}_1 = \vec{0}$, $\vec{V}_3 = 2\vec{V}_1$ LD

$\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ LI

$\begin{bmatrix} 1 & \cancel{1} \\ 1 & \cancel{1} \\ 0 & \cancel{0} \end{bmatrix}$ LD $\vec{V}_2 = \vec{V}_1$

1.9.1 Write down a matrix that sends $\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to $\vec{\mathbf{b}}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but also sends $\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to $\vec{\mathbf{b}}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\vec{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to $\vec{\mathbf{b}}_3 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$.

For Thursday