1.7: Linear Dependence MA322-007 Feb 5 Worksheet

A **linear dependence relation** amongst vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$ is a sequence of numbers $c_1, c_2, \ldots, c_n$ such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \ldots + c_n\vec{\mathbf{v}}_n = \vec{\mathbf{0}}$.

If A is a matrix whose columns are $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$, and $\vec{\mathbf{c}}$ is a vector whose entries are $c_1, \ldots, c_n$, then $A\vec{\mathbf{c}} = c_1\vec{\mathbf{v}}_1 + \ldots c_n\vec{\mathbf{v}}_n$.

A linear dependence relation $\vec{\mathbf{c}}$ amognst the columns of A is exactly a homogeneous solution $A\vec{\mathbf{c}} = \vec{\mathbf{0}}$.

Linear dependence amongst the columns of A means there are infinitely many solutions to each $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ for which there is at least one solution. Linear independence amongst the columns of A means there is at most one solution.

A linear dependency relation amongst the rows of A requires the same relation to hold in the rows (entries) of $\vec{\mathbf{b}}$ for there to be a solution to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Row reduction finds these dependencies when it finds zero rows. The number of rows of A is equal to number of zero rows in the RREF of A plus the number of linearly independent columns. Each linearly independent column creates a new direction for the image $\vec{\mathbf{b}}$, and so each missing direction creates a new requirement on $\vec{\mathbf{b}}$.

1.8: Linear transformations

This function $A(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ is **linear**: $A(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = A(\vec{\mathbf{x}}) + A(\vec{\mathbf{y}})$ and $A(c\vec{\mathbf{x}}) = cA(\vec{\mathbf{x}})$.

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. What does it do to a vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$? $A\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1+x_2 \end{bmatrix}$$

What does A do the vectors $\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is just $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $A\vec{e}_2$ is always the 2nd column of A

What must it do to $x_1\vec{\mathbf{e}}_1 + x_2\vec{\mathbf{e}}_2$?

$$A(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1 A(\vec{e}_1) + x_2 A(\vec{e}_2) = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1+x_2 \end{bmatrix}.$$

No Surprise tho. $x_1\vec{e}_1 + x_2\vec{e}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Does $A(\vec{\mathbf{x}}) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ have a solution?

No. $3+4 \neq 5$ but $R_1 + R_2 = R_3$ in A.

Takeaway: $\vec{\mathbf{x}} \xrightarrow{A} \vec{\mathbf{b}}$

MA322-007 Feb 5 quiz Name: Key

HW1.7 #13 For what values of h are the columns of A linearly independent? Explain why.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 9 & h \\ 3 & 6 & 9 \end{bmatrix}.$$ ⟵ Missing negative signs. Actual #13 on next sheet

Row Reduce and check for free variable.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 5 & 9 & h & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1-2\,\text{new}R_2]{R_2/-1} \left[\begin{array}{ccc|c} 1 & 0 & 2h-27 & 0 \\ 0 & 1 & 15-h & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $c_1 = (27-2h)c_3$, $c_2 = (h-15)c_3$, c_3 is free are the linear depence relations for <u>any</u> h.

No matter what h is, $(2h-27)\vec{v}_1 + (15-h)\vec{v}_2 = \vec{v}_3$ shows the columns are dependent. $(c_3=-1)$

1.9.1 Write down a matrix that sends $\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to $\vec{\mathbf{b}}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but also sends $\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to $\vec{\mathbf{b}}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\vec{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to $\vec{\mathbf{b}}_3 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$.

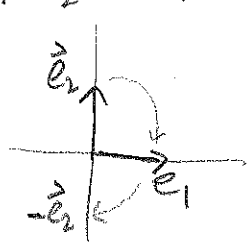
$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$ is the only correct answer

1.9.2 Write down a matrix that rotates vectors by 90° clockwise around the origin. Hint: What does such a matrix do to $\vec{\mathbf{i}} = \vec{\mathbf{e}}_1$ and $\vec{\mathbf{j}} = \vec{\mathbf{e}}_2$.

$A\vec{e}_1 = -\vec{e}_2$

$A\vec{e}_2 = \vec{e}_1$

$$A = \begin{bmatrix} \uparrow & \uparrow \\ -e_2 & e_1 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



HW 1.7 #13 $\begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{bmatrix}$

$\qquad\qquad\quad \vec{v}_1 \quad \vec{v}_2 \quad v_3$

Note $\vec{v}_1 - \vec{v}_2 = \vec{v}_3$ (On the quiz, I wanted $\vec{v}_1 + \vec{v}_2 = \vec{v}_3$)

if $h = 14$

So $h = 14$ is *definitely linearly dependent.*

Any other h? Row Reduce!

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{array}\right] \xrightarrow[R_3 + 3R_1]{R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2h-27 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = (27 - 2h)x_3$ Take $x_3 = -1$ $x_1 = 2h - 27$

$x_2 = (15 - h)x_3$ $x_2 = h - 15$

x_3 is free $x_3 = -1$

$$x_1 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(2h-27)\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} + (h-15)\begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ h \\ 9 \end{bmatrix}$$

so <u>all</u> h are linearly dependent. $h = 14$ gives $1\vec{v}_1 - 1\vec{v}_2 = \vec{v}_3$

but $h = 15$ gives $3\vec{v}_1 = \vec{v}_3$

and $h = 13$ gives $-\vec{v}_1 - 2\vec{v}_2 = \vec{v}_3$

etc