1.8: Matrices are functions

MA322-007 Feb 10 Worksheet

An $R \times C$ matrix A with R rows and C columns is also a function whose domain is vectors $\vec{\mathbf{c}}, \vec{\mathbf{x}}$ of size C and whose codomain is vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_C, \vec{\mathbf{b}}$ of size R. For example $A\vec{\mathbf{c}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \ldots + c_C\vec{\mathbf{v}}_C$ expresses the output of $A(\vec{\mathbf{c}})$ as a linear combination of the vectors $\vec{\mathbf{v}}_i$ that are the columns of A.

Not all functions from vectors to vectors are matrices: only the ones that satisfy the two axioms: $A(r\vec{\mathbf{c}}) = r \cdot A(\vec{\mathbf{c}})$ and $A(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = A(\vec{\mathbf{x}}) + A(\vec{\mathbf{y}})$ for all vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ in the domain and all scalars r.

1.9: Blackbox matrix

The vectors $\vec{\mathbf{e}}_i$ that have a 1 in the ith position and a 0 elsewhere are called the **standard basis vectors**. If $A(\vec{\mathbf{e}}_i) = \vec{\mathbf{v}}_i$, then $\vec{\mathbf{v}}_i$ is exactly the ith column of A. In order to find the columns of A when A is only described in words, just calculate $A(\vec{\mathbf{e}}_i)$, the image of each standard basis vector under the action of A. By the two matrix axioms, if $\vec{\mathbf{x}} = x_1\vec{\mathbf{e}}_1 + \ldots + x_n\vec{\mathbf{e}}_n$, then $A(\vec{\mathbf{x}}) = x_1\vec{\mathbf{v}}_1 + \ldots + x_n\vec{\mathbf{v}}_n$.

2.1: Matrix operations

If we can take linear combinations of elements of the codomains, we can take linear combinations of the functions. If A, B are matrices of the same size and r is a scalar, then define $A+B$ to be the matrix of the same size that takes $\vec{\mathbf{x}}$ to $A(\vec{\mathbf{x}}) + B(\vec{\mathbf{x}})$ and rA to be the matrix of the same size that takes $\vec{\mathbf{x}}$ to $r(A(\vec{\mathbf{x}}))$. We can check that this makes $A + B$ and rA into matrices.

p93 Example 1,2: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$.

$A+B$ Just Add the Columns!

$\begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$

$A+C$ Not enough columns! Domain Mismatch!

$2B$ Just Multiply the Columns!

$\begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$

$A-2B$ Just Linear Combinate the columns!

$\begin{bmatrix} 4-2 & 0-2 & 5-2 \\ -1-6 & 3-10 & 2-14 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$

2.1.b: Matrices are vectors

We've seen different types of vectors. 2D vectors. 3D vectors. Chemical vectors. Each type of vector obeys the same basic rules: you can add and subtract vectors and multiply them by numbers. If $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are vectors of the same type and r, s are numbers, then (1) $(r+s)\vec{\mathbf{a}} = r\vec{\mathbf{a}} + s\vec{\mathbf{a}}$, (2) $r(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = r\vec{\mathbf{a}} + r\vec{\mathbf{b}}$, (3) $1\vec{\mathbf{a}} = \vec{\mathbf{a}}$, (4) $r(s\vec{\mathbf{a}}) = (rs)\vec{\mathbf{a}}$, (5) $0\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}}$, and (6) $\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$. We abbreviate $0\vec{\mathbf{a}}$ as $\vec{\mathbf{0}}$ and $(-1)\vec{\mathbf{a}}$ as $-\vec{\mathbf{a}}$, and we write $\vec{\mathbf{b}} + (-1)\vec{\mathbf{a}}$ as $\vec{\mathbf{b}} - \vec{\mathbf{a}}$.

Now we have a new type of vector for every pair of positive integers R, C: the $R \times C$ matrices are vectors where we can form linear combinations.

2.1.c: Matrix multiplication We define the composition of two matrices AB by the rule $(AB)\vec{\mathbf{x}} = A(B\vec{\mathbf{x}})$. This requires that codomain of B be the domain of A, that is, if A is $m \times n$ and B is $n' \times p$, then we must have $n = n'$.

Suppose B has columns $\vec{\mathbf{b}}_1$, $\vec{\mathbf{b}}_2$, ..., $\vec{\mathbf{b}}_p$ all of which are in the domain of A. Then AB has columns $A\vec{\mathbf{b}}_1$, $A\vec{\mathbf{b}}_2$, ..., $A\vec{\mathbf{b}}_p$. In particular, AB has the same number of columns as B does.

$$\text{If } B = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\mathbf{b}}_1 & \vec{\mathbf{b}}_2 & \dots & \vec{\mathbf{b}}_p \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}, \text{ then } AB = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ A\vec{\mathbf{b}}_1 & A\vec{\mathbf{b}}_2 & \dots & A\vec{\mathbf{b}}_p \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$\text{If } \vec{\mathbf{b}}_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{1n} \end{bmatrix} \text{ and } A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}, \text{ then } A\vec{\mathbf{b}}_1 = b_{11}\vec{\mathbf{a}}_1 + b_{12}\vec{\mathbf{a}}_2 + \ldots + b_{1n}\vec{\mathbf{a}}_n.$$

These satisfy the additional axioms similar to numbers (most of which follow from being functions that output vectors): $A(BC) = (AB)C$, $(A+B)C = AC + BC$, $A(B+C) = AB + AC$, $x(AB) = (xA)B = A(xB)$. Additionally the matrix I_m consisting of the m different m-D standard basis vectors acts like 1, $I_m A = A = AI_n$ if A is $m \times n$.

p95 Example 3: Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$ $\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \neq \mathbb{R}^3 \xrightarrow{B} \mathbb{R}^2$

AB *BA*

Undefined $A\vec{x}$ is not in the domain of B

$$\left[\, A\begin{bmatrix}4\\1\end{bmatrix} \,\middle|\, A\begin{bmatrix}3\\-2\end{bmatrix} \,\middle|\, A\begin{bmatrix}6\\3\end{bmatrix} \,\right]$$

$$\left[\, 4\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}3\\-5\end{bmatrix} \,\middle|\, 3\begin{bmatrix}2\\1\end{bmatrix}-2\begin{bmatrix}3\\-5\end{bmatrix} \,\middle|\, 6\begin{bmatrix}2\\1\end{bmatrix}+3\begin{bmatrix}3\\-5\end{bmatrix} \,\right] = \left[\begin{array}{c|c|c} 11 & 0 & 21 \\ -1 & 13 & -9 \end{array}\right]$$

Example 7: Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

p98 *AB* *BA*

$$\left[\, A\begin{bmatrix}2\\4\end{bmatrix} \,\middle|\, A\begin{bmatrix}0\\3\end{bmatrix} \,\right] \neq \left[\, B\begin{bmatrix}5\\3\end{bmatrix} \,\middle|\, B\begin{bmatrix}1\\-2\end{bmatrix} \,\right]$$

$$\left[\begin{array}{c|c} 14 & 3 \\ -2 & -6 \end{array}\right] \neq \left[\begin{array}{c|c} 10 & 2 \\ 29 & -2 \end{array}\right]$$

See exercise 10 and 12 for more extreme examples.

2.1.d: Square matrices See exercise 11 for an important square example. Also I_n.

2.1.e: Transpose See exercise 27 for an important construction.

MA322-007 Feb 10 quiz Name: Key

HW1.9 #1 If $A(\vec{e}_1) = (3,1,3,1)$ and $A(\vec{e}_2) = (-5,2,0,0)$, then what is the matrix of A, assuming A is a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^4$ and $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$.

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

HW1.9 #15 Find the matrix of A where $A(x,y,z) = (2x-4y, x-z, -y+3z)$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$.

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-4y \\ x-z \\ -y+3z \end{bmatrix} = x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \quad \text{so } A = \begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

HW1.9 #17 Find the matrix of A where $A(a,b,c,d) = (a+2b, 0, 2b+d, b-d)$ and $A : \mathbb{R}^4 \to \mathbb{R}^4$.

$$A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{so } A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

2.1 (#11a) Find the matrix of D where $D(x,y,z) = (5x, 3y, 2z)$.

$$D\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad \text{so } D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2.1 (#11b) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, find the matrix of DA.

$$DA = \begin{bmatrix} 5 & 10 & 15 \\ 6 & 12 & 15 \\ 6 & 10 & 12 \end{bmatrix}$$

or longer $\left[1\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \;\middle|\; \cdots \right]$

also try $AD = \left[5\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \;\middle|\; 3\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \;\middle|\; 2\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \right]$