2.1: Matrices are functions MA322-007 Feb 12 Worksheet

An $R \times C$ matrix A with R rows and C columns is a function whose domain is vectors $\vec{\mathbf{c}}, \vec{\mathbf{x}}$ of size C and whose codomain is vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_C, \vec{\mathbf{b}}$ of size R.

We can take linear combinations of functions as long as (a) their domains overlap, and (b) their codomains consist of vectors. For example, if r, s are numbers and A, B are matrices, then $rA + sB$ is the function that takes $\vec{\mathbf{x}}$ to $r(A(\vec{\mathbf{x}})) + s(B(\vec{\mathbf{x}}))$. We can multiply functions as long as the codomain of one is the domain of the other. For example if A has domain vectors of size n and B has codomain vectors of size n, then AB is the function with the same domain as B and the same codomain as A and it takes $\vec{\mathbf{x}}$ from the domain of B to $A(B(\vec{\mathbf{x}}))$ in the codomain of A.

If A, B are matrices and r, s are numbers, then $rA+sB$ is (either undefined or) a matrix, and AB is (either undefined or) a matrix. This is because they satisfy the two matrix axioms: $T(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = T(\vec{\mathbf{x}}) + T(\vec{\mathbf{y}})$ and $T(r\vec{\mathbf{x}}) = r(T(\vec{\mathbf{x}}))$.

2.1.d: Square matrices have equal domain and codomain. If A and B are $n \times n$ matrices, then so are rA, A^n for non-nonegative n, $A + B$, and AB. This means we can form polynomials in square matrices. If we are careful we can form power series, and so define things like $\sin(A)$, $\exp(A)$, and $\ln(1 + A)$.

A very nice kind of matrix is a diagonal matrix. It has numbers d_1, d_2, $\ldots$, d_n so that $D(\vec{\mathbf{e}}_i) = d_i\vec{\mathbf{e}}_i$. In other words, it just multiplies each entry by a number (it can be a different number for each entry).

The diagonal matrix where every number d_i is 1 is called the identity matrix, I_n. It multiplies everything by 1, so $I_n\vec{\mathbf{v}} = \vec{\mathbf{v}}$. The zero matrix 0_n is the diagonal matrix with each $d_i = 0$. It multiplies everything by 0, so $0_n\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

2.2: Inverse matrices We've found linear combinations of matrices, and multiplied matrices. Now we'd like to divide matrices.

A^{-1} is the matrix with the property that if $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, then $A^{-1}(\vec{\mathbf{b}}) = \vec{\mathbf{x}}$. By applying A^{-1} to the standard basis vectors we see that $A^{-1}A = I_n$. Such a matrix is called a **left inverse** matrix of A.

Math textbooks (such as ours) usually require that $AA^{-1} = I_n$ as well. Such matrices are called **inverse** matrices. If A is square, then this happens automatically (a left inverse matrix of a square matrix is automatically a right inverse matrix).

Suppose $A = [\ 3\ ]$. Find A^{-1}.

$$A^{-1} = \left[\ 1/3\ \right]$$

Do all 1×1 matrices have an inverse? [Write down the rule.]

$$\text{If } A = [a], \text{ then } A^{-1} \begin{cases} = [1/a] & \text{if } a \neq 0 \\ \text{DNE} & \text{if } a = 0 \end{cases}$$

What about $A = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Can you find a left inverse of A? Hint: What does $\vec{\mathbf{x}}$ look like? If you know $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and you know the first entry of $\vec{\mathbf{b}}$, how do you find $\vec{\mathbf{x}}$?

$A\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}[x] = \begin{bmatrix} 3x \\ 5x \end{bmatrix}$ so $A^{-1}\begin{bmatrix} 3x \\ 5x \end{bmatrix} = [x]$ $A^{-1} = [\tfrac{1}{3} \;\; 0]$ or $A^{-1} = [0 \;\; \tfrac{1}{5}]$

How many left inverses are there?

∞ $A^{-1} = [a \;\; b]$ where $3a + 5b = 1$

$A^{-1}A = [a \;\; b]\begin{bmatrix} 3 \\ 5 \end{bmatrix} = [3a+5b] = [1]$

Are any of them right inverses?

No. $AA^{-1} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}[a \;\; b] = \begin{bmatrix} 3a & 3b \\ 5a & 5b \end{bmatrix}$ if $3a=1$ then $5a \neq 0$ not I_2

What about $A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$. Can you find a left inverse of A? Hint: Same hint, but different answers.

$A\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 5y \end{bmatrix}$ $A^{-1}\begin{bmatrix} 3x \\ 5y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $A^{-1} = \begin{bmatrix} \tfrac{1}{3} & 0 \\ 0 & \tfrac{1}{5} \end{bmatrix}$

How many left inverses are there?

Just one (hard, follows from RREF) or from $A^{-1}A = AA^{-1}$ in this case

Are any of them right inverses?

Yes. All/The only one works.

Can you give an example of 2×2 matrix that cannot have either a left or a right inverse?

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

What is the rule for 2×2 matrices?

See Book ☺ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} \begin{cases} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \text{DNE if } ad-bc=0 \end{cases}$

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HW2.1 #11a If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then compute AD and DA (label them clearly).

$$AD = \left[A\begin{bmatrix}5\\0\\0\end{bmatrix} \middle| A\begin{bmatrix}0\\3\\0\end{bmatrix} \middle| A\begin{bmatrix}0\\0\\2\end{bmatrix} \right] = \left[\begin{array}{c|c|c} 5(1) & 3(2) & 2(3) \\ 5(2) & 3(4) & 2(5) \\ 5(3) & 3(5) & 2(6) \end{array}\right]$$

$$DA = \left[\begin{array}{c|c|c} 5(1) & 5(2) & 5(3) \\ 3(2) & 3(4) & 3(5) \\ 2(3) & 2(5) & 2(6) \end{array}\right]$$

HW2.1 #11b Describe in words what D does to the column or rows of A when you do AD.

AD takes the first column of A and multiplies it by 5
2nd column by 3
3rd column by 2

DA is similar but rows

HW2.1 #11c Find a matrix B so that $AB = BA$, but B is neither the zero matrix nor the identity matrix.

$B = \begin{bmatrix} a & & \\ & a & \\ & & a \end{bmatrix}$ is ~~clearly~~ the easiest. As long as $a \neq 0, 1$ then $B \neq 0, I$.

AB multiplies columns by a, BA multiplies rows by a.

HW2.1 #12 Find a matrix B so that AB is the zero matrix, but B has NO zero entries. Here $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$.

$\begin{array}{cc} \vec{v}_1 & \vec{v}_2 \end{array}$

$2\vec{v}_1 + \vec{v}_2 = \vec{0}$ so $A\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ works

$4\vec{v}_1 + 2\vec{v}_2 = \vec{0}$ $A\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $B = \begin{bmatrix} 2s & 2t \\ s & t \end{bmatrix}$ works

So $A\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Weirdly (or not) $BA = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ is not 0!

Circle one of the next two questions and answer it (first is easier, second was homework):

HW2.1 #20 Suppose the first two columns $\vec{b}_1$ and $\vec{b}_2$ of B are equal. What can you conclude about the columns of AB? OR

HW2.1 #21 Suppose the last column of AB is entirely zero, but B has no columns that are entirely zero. What can you conclude about the columns of A?

#20 $AB = [A\vec{b}_1 \mid A\vec{b}_2 \mid \cdots]$

if $\vec{b}_1 = \vec{b}_2$, then $A\vec{b}_1 = A\vec{b}_2$

so the first two columns of AB are Equal

#21 $AB = [\cdots \mid A\vec{b}_n]$

So $A\vec{b}_n = \vec{0}$ but $\vec{b}_n \neq 0$, so $\vec{b}_n$ is a nontrivial linear dependence relation amongst the columns of A: $b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots b_m\vec{v}_m$

2.2.1 For $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find D^{-1}.

$D\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7x \\ 5y \\ 3z \end{bmatrix}$

$D^{-1}\begin{bmatrix} 7x \\ 5y \\ 3z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

So $D^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a/7 \\ b/5 \\ c/3 \end{bmatrix}$

$D^{-1} = \begin{bmatrix} 1/7 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

2.2.2 For $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, find D^{-1}.

$D\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7x \\ 5y \\ 0 \end{bmatrix}$

$D^{-1}\begin{bmatrix} 7x \\ 5y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z? \end{bmatrix}$

How can we find z from 0?

Can't. D^{-1} $\boxed{\text{DNE}}$

Closest is $D^{-1} = \begin{bmatrix} \frac{1}{7} & 0 & 0 \\ 0 & 1/5 & 0 \\ \star & \star & \star \end{bmatrix}$

where $\star\star\star$ could be anything (because it is wrong) but 0 0 0 is the smallest possibility