2.1: Matrices are functions MA322-007 Feb 12 Worksheet

An $R \times C$ matrix A with R rows and C columns is a function whose domain is vectors $\vec{\mathbf{c}}, \vec{\mathbf{x}}$ of size C and whose codomain is vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_C, \vec{\mathbf{b}}$ of size R.

We can take linear combinations of functions as long as (a) their domains overlap, and (b) their codomains consist of vectors. For example, if r, s are numbers and A, B are matrices, then $rA + sB$ is the function that takes $\vec{\mathbf{x}}$ to $r(A(\vec{\mathbf{x}})) + s(B(\vec{\mathbf{x}}))$. We can multiply functions as long as the codomain of one is the domain of the other. For example if A has domain vectors of size n and B has codomain vectors of size n, then AB is the function with the same domain as B and the same codomain as A and it takes $\vec{\mathbf{x}}$ from the domain of B to $A(B(\vec{\mathbf{x}}))$ in the codomain of A.

If A, B are matrices and r, s are numbers, then $rA+sB$ is (either undefined or) a matrix, and AB is (either undefined or) a matrix. This is because they satisfy the two matrix axioms: $T(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = T(\vec{\mathbf{x}}) + T(\vec{\mathbf{y}})$ and $T(r\vec{\mathbf{x}}) = r(T(\vec{\mathbf{x}}))$.

2.1.d: Square matrices have equal domain and codomain. If A and B are $n \times n$ matrices, then so are rA, A^n for non-nonegative n, $A + B$, and AB. This means we can form polynomials in square matrices. If we are careful we can form power series, and so define things like $\sin(A)$, $\exp(A)$, and $\ln(1 + A)$.

A very nice kind of matrix is a diagonal matrix. It has numbers d_1, d_2, $\ldots$, d_n so that $D(\vec{\mathbf{e}}_i) = d_i\vec{\mathbf{e}}_i$. In other words, it just multiplies each entry by a number (it can be a different number for each entry).

The diagonal matrix where every number d_i is 1 is called the identity matrix, I_n. It multiplies everything by 1, so $I_n\vec{\mathbf{v}} = \vec{\mathbf{v}}$. The zero matrix 0_n is the diagonal matrix with each $d_i = 0$. It multiplies everything by 0, so $0_n\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

2.2: Inverse matrices We've found linear combinations of matrices, and multiplied matrices. Now we'd like to divide matrices.

A^{-1} is the matrix with the property that if $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, then $A^{-1}(\vec{\mathbf{b}}) = \vec{\mathbf{x}}$. By applying A^{-1} to the standard basis vectors we see that $A^{-1}A = I_n$. Such a matrix is called a **left inverse** matrix of A.

Math textbooks (such as ours) usually require that $AA^{-1} = I_n$ as well. Such matrices are called **inverse** matrices. If A is square, then this happens automatically (a left inverse matrix of a square matrix is automatically a right inverse matrix).

Suppose $A = \left[\ 3\ \right]$. Find A^{-1}.

Do all 1×1 matrices have an inverse? [Write down the rule.]

What about $A = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Can you find a left inverse of A? Hint: What does $\vec{\mathbf{x}}$ look like? If you know $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and you know the first entry of $\vec{\mathbf{b}}$, how do you find $\vec{\mathbf{x}}$?

How many left inverses are there?

Are any of them right inverses?

What about $A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$. Can you find a left inverse of A? Hint: Same hint, but different answers.

How many left inverses are there?

Are any of them right inverses?

Can you give an example of 2×2 matrix that cannot have either a left or a right inverse?

What is the rule for 2×2 matrices?

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HW2.1 #11a If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then compute AD and DA (label them clearly).

HW2.1 #11b Describe in words what D does to the column or rows of A when you do AD.

HW2.1 #11c Find a matrix B so that $AB = BA$, but B is neither the zero matrix nor the identity matrix.

HW2.1 #12 Find a matrix B so that AB is the zero matrix, but B has NO zero entries. Here $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$.

Circle one of the next two questions and answer it (first is easier, second was homework):

HW2.1 #20 Suppose the first two columns $\vec{\mathbf{b}}_1$ and $\vec{\mathbf{b}}_2$ of B are equal. What can you conclude about the columns of AB? OR

HW2.1 #21 Suppose the last column of AB is entirely zero, but B has no columns that are entirely zero. What can you conclude about the columns of A?

2.2.1 For $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find D^{-1}.

2.2.2 For $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, find D^{-1}.