MA322-007 Feb 24 Exam Name: Key

Explain your answers, briefly, on each page.
Numbers without justification are worth no credit.

1. Perform the following arithmetic operations or explain why they are not defined.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 100 \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$

Add Columns, but mismatched sizes

DNE

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 100 \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

Add Columns

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] + \left[\begin{array}{c|c|c} 700 & 800 & 900 \\ 1000 & 1100 & 1200 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 701 & 802 & 903 \\ 1004 & 1105 & 1206 \end{array}\right]$$

(c) $\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$\left[\begin{array}{c|c} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} & \cdots \end{array}\right] = \left[\begin{array}{c|c} 1\begin{bmatrix} 4 \\ 7 \end{bmatrix} + 4\begin{bmatrix} 5 \\ 8 \end{bmatrix} + ?\begin{bmatrix} 6 \\ 9 \end{bmatrix} & \cdots \end{array}\right]$$

size mismatch DNE

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 & 8 & 9 \\ 10 & 20 & 0 \end{bmatrix}$

$$\left[\begin{array}{c|c} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 10 \end{bmatrix} & \cdots \end{array}\right] = \left[\begin{array}{c|c} 7\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} & \cdots \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} \begin{matrix} 7 \\ 21 \\ 35 \end{matrix} + \begin{matrix} 20 \\ 40 \\ 60 \end{matrix} & \begin{matrix} 8 \\ 24 \\ 40 \end{matrix} + \begin{matrix} 40 \\ 80 \\ 120 \end{matrix} & \begin{matrix} 9 \\ 27 \\ 45 \end{matrix} + \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 27 & 45 & 9 \\ 61 & 104 & 27 \\ 95 & 160 & 45 \end{array}\right]$$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ a & 0 & 1 \end{bmatrix}^{-1}$ (for a an arbitrary real number)

$\mathrm{RREF}\,(A|I) = (I|A^{-1})$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ a & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2/2]{R_3 - aR_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -a & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ -a & 0 & 1 \end{bmatrix}$$

Notice Diagonal is Multiplicative Inverse $\left(\frac{\circ}{\circ}\right)$

and off-diagonal is Additive Invers $(-)$

2. Convert between a description of a linear transformation and its matrix.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, $T(\vec{\mathbf{e}}_1) = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, $T(\vec{\mathbf{e}}_2) = \left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$, and $T(\vec{\mathbf{e}}_3) = \left[\begin{smallmatrix}5\\6\end{smallmatrix}\right]$. What is $T(\vec{\mathbf{v}})$ if $\vec{\mathbf{v}} = \vec{\mathbf{e}}_1 + 100\vec{\mathbf{e}}_2$?

$$T(\vec{e}_1 + 100\vec{e}_2) = T(\vec{e}_1) + 100\,T(\vec{e}_2) \to \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}300\\400\end{bmatrix}$$

$$= \begin{bmatrix}301\\402\end{bmatrix}$$

(b) Find a matrix A such that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}} \in \mathbb{R}^3$, where T is from part (a).

$$A = \left[\, T(\vec{e}_1) \mid T(\vec{e}_2) \mid T(\vec{e}_3) \,\right] = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$$

(c) If $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation defined by $S(\vec{\mathbf{x}}) = B\vec{\mathbf{x}}$ for $B = \begin{bmatrix} 1 & 10 & 100 \\ 2 & 20 & 200 \end{bmatrix}$, what is $S(\vec{\mathbf{e}}_3)$?

$$S(\vec{e}_3) = B\begin{bmatrix}0\\0\\1\end{bmatrix} = 0\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}10\\20\end{bmatrix} + 1\begin{bmatrix}100\\200\end{bmatrix}$$

$$= \begin{bmatrix}100\\200\end{bmatrix}$$

(d) If $F : \mathbb{R}^3 \to \mathbb{R}^3$ switches the $\vec{\mathbf{i}}$ and $\vec{\mathbf{k}}$ components (the "x" and "z" components) then find a matrix C so that $F(\vec{\mathbf{v}}) = C\vec{\mathbf{v}}$ for every $\vec{\mathbf{v}}$ in $\mathbb{R}^3$. **Hint:** What is $F(\vec{\mathbf{e}}_1)$?

$$C = \left[\, F(\vec{e}_1) \mid F(\vec{e}_2) \mid F(\vec{e}_3) \,\right] = \left[\, \vec{e}_3 \mid \vec{e}_2 \mid \vec{e}_1 \,\right]$$

$$= \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

3. For $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

Find a matrix A satisfying the given properties:

(a) $A^2 = I_2$ but $A \neq \pm I_n$ (Challenge: A has no zero entries)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+dd \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$c(a+d)=0$
$b(a+d)=0$
if no nonzero, then $b \neq 0 \neq c$ so $a+d=0$

$d=-a$ so $a^2+bc=1$, $d^2+bc=1$

$b=1$
$c=-3$
$a^2=4$
$a=2$
$d=-2$

$\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$

(b) $AB = 0$ but $BA \neq 0$ (Challenge: find all A)

$AB = \left[A\begin{bmatrix}1\\1\\-1\end{bmatrix} \middle| A\begin{bmatrix}1\\1\\-1\end{bmatrix} \middle| A\begin{bmatrix}1\\-1\\0\end{bmatrix} \right]$ if $A = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3]$

$= [\vec{v}_1+\vec{v}_2-\vec{v}_3 \,|\, \text{ditto} \,|\, \vec{v}_1-\vec{v}_2] = [0|0|0]$

so $\vec{v}_1+\vec{v}_2 = \vec{v}_3$, $\vec{v}_1 = \vec{v}_2$, so $A = \begin{bmatrix} \pi & \pi & 2\pi \\ \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 73 & 73 & 146 \end{bmatrix} = \begin{bmatrix} a & a & 2a \\ b & b & 2b \\ c & c & 2c \end{bmatrix}$

(c) $CA = I_2$ (Challenge: find all A)

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} = \begin{bmatrix} x+3z & a+3c \\ 2y & 2b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x+3z=1$
$2y=0$

$a+3c=0$
$2b=1$

$A = \begin{bmatrix} 1-3z & -3c \\ 0 & 1/2 \\ z & c \end{bmatrix}$ for example $\begin{bmatrix} -14 & +39 \\ 0 & 1/2 \\ 5 & -13 \end{bmatrix}$

(d) $AD = DA$ but $A \neq 0$, $A \neq I_3$ (Challenge: find all A)

$\left[A\begin{bmatrix}7\\0\\0\end{bmatrix} \middle| A\begin{bmatrix}0\\8\\0\end{bmatrix} \middle| A\begin{bmatrix}0\\0\\9\end{bmatrix} \right] = [7\vec{v}_1 | 8\vec{v}_2 | 9\vec{v}_3]$

but DA is by rows $\begin{bmatrix} 7r_1 \longrightarrow \\ 8r_2 \longrightarrow \\ 9r_3 \longrightarrow \end{bmatrix}$

so if $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $7d = 8d \Rightarrow d = 0$. All off diagonals $=0$

but $7a = 7a \Rightarrow a$ is free

$A = \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}$ for example $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. For each matrix A explain why it is invertible or not.

(a) $\begin{array}{c} \begin{bmatrix} 3 & 0 & 3 \\ 2 & 0 & 2 \\ 2 & 7 & 9 \end{bmatrix} \\ \;v_1 \;\; v_2 \;\; v_3 \end{array}$ $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$ Lin Dep so not INV

Not

(b) $\begin{bmatrix} 3 & 2 & 2 \\ 0 & 0 & 7 \\ 3 & 2 & 8 \end{bmatrix}$ $2\vec{v}_1 = 3\vec{v}_2$ Lin Dep so Not INV

Not

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$ Has one pivot in each column so $RREF(A|I) = (I|A^{-1})$ will work

INV

(d) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ x_1 is free so not invertible $(RREF(A|I) \neq (I|-))$

$\vec{v}_1 = 0$ so Lin Dep so not INV

(e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Inv

$(x, y, z) \xrightarrow{A} (y, z, x) \xrightarrow{A^{-1}} (x, y, z)$

$A\vec{e}_1 = \vec{e}_3$, $A\vec{e}_2 = \vec{e}_1$, $A\vec{e}_3 = \vec{e}_2$ so $A^{-1}\vec{e}_1 = \vec{e}_2$, $A^{-1}\vec{e}_2 = \vec{e}_3$, $A^{-1}\vec{e}_3 = \vec{e}_1$

$\left(\frac{m}{x}, \frac{m}{y}, \frac{m}{z}\right) \longrightarrow \left(\frac{m}{z}, \frac{m}{x}, \frac{m}{y}\right)$

$A^{-1} \begin{array}{c} \vec{e}_2 \;\; \vec{e}_3 \;\; \vec{e}_1 \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$