MA322-007 Feb 24 Practice Exam Name: Key

Explain your answers, briefly, on each page.
Numbers without justification are worth no credit.

1. Perform the following arithmetic operations or explain why they are not defined.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 10 & 20 \\ 30 & 40 \\ 50 & 60 \end{bmatrix} = \begin{bmatrix} 11 & 22 \\ 33 & 44 \\ 55 & 66 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 10 \begin{bmatrix} 7 & 8 & 9 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 70 & 80 & 90 \\ 70 & 80 & 90 \end{bmatrix}$

$= \begin{bmatrix} 71 & 82 & 93 \\ 74 & 85 & 96 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ Not defined, dimensions do not match

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \left(3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}$

Not defined these dimensions do not match

(e) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \left(3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2\cdot 6 \\ 3\cdot 3 + 4\cdot 6 \\ 5\cdot 3 + 6\cdot 6 \end{bmatrix} = \begin{bmatrix} 3+12 \\ 9+24 \\ 15+36 \end{bmatrix} = \begin{bmatrix} 15 \\ 33 \\ 51 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & a & a \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$ (for a an arbitrary real number)

$\begin{bmatrix} 1 & a & a \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$b + a = 0 \Rightarrow b = -a$

$c + da + a = 0 \Rightarrow c = -a - da$
$= -a + a = 0$

$d + 1 = 0 \Rightarrow d = -1$

So the inverse is

$\begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

2. Convert between a description of a linear transformation and its matrix.

(a) If $T : \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation satisfying $T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T(\vec{\mathbf{e}}_2) = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

then what is $T(\vec{\mathbf{v}})$ if $\vec{\mathbf{v}} = 10\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2$?

$$T(\vec{v}) = 10 \cdot T(\vec{e}_1) + T(\vec{e}_2)$$
$$= 10\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 15 \\ 26 \\ 37 \\ 48 \end{bmatrix}$$

(b) Find a matrix A such that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}} \in \mathbb{R}^2$, where T is from part (a).

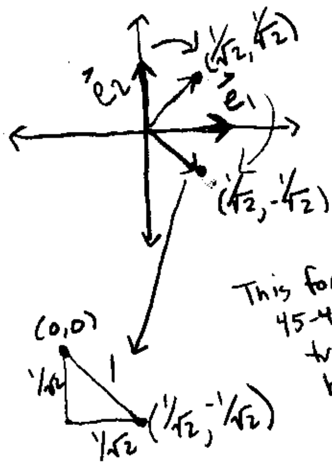
$$A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

(c) If $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation defined by $S(\vec{\mathbf{x}}) = B\vec{\mathbf{x}}$ for $B = \begin{bmatrix} 1 & 10 & 100 \\ 2 & 20 & 200 \end{bmatrix}$, what is $S(\vec{\mathbf{e}}_2)$?

$$S(\vec{e}_2) = B\vec{e}_2 = \begin{bmatrix} 1 & 10 & 100 \\ 2 & 20 & 200 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

(d) If $R : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors 45 degrees clockwise, then find a matrix C so that $R(\vec{\mathbf{v}}) = C\vec{\mathbf{v}}$ for every $\vec{\mathbf{v}}$ in $\mathbb{R}^2$. **Hint:** What is $R(\vec{\mathbf{e}}_1)$?

We can determine the matrix by figuring out where it would take $\vec{e}_1$ and $\vec{e}_2$



$$R(\vec{e}_1) = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \quad R(\vec{e}_2) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

The matrix that performs this linear transformation is $C = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

This forms a 45-45-90 triangle with hypotenuse 1

3. For $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

Find a matrix A satisfying the given properties:

(a) $A^2 = 0$ but $A \neq 0$ (Challenge: A has no zero entries)

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) $AB = 0$ but $BA \neq 0$ (Challenge: find all A)

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \neq 0$$

(c) $AC = CA + \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ (Challenge: find all A)

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad AC = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$CA + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

(d) $AD = DA$ but $A \neq 0$, $A \neq I_3$ (Challenge: find all A)

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad AD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$DA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

(e) $EA = I_2$ (Challenge: find all A)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 5 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. For each matrix A explain why it is invertible or not.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 6 & 7 & 6 \end{bmatrix}$ Not invertible; It has two identical columns.

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ Not invertible; It has two identical rows.

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 6 & 7 & 0 \end{bmatrix}$ Not invertible; It has a zero column.

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ Invertible! RREF pivots in all columns.

(e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Invertible! Rows can be rearranged to form the Identity matrix, which has itself as an inverse.

Also, this matrix is its own inverse!

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$