MA322-007 Feb 24 Practice Exam Name: Key 2

Explain your answers, briefly, on each page.
Numbers without justification are worth no credit.

1. Perform the following arithmetic operations or explain why they are not defined.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 10 & 20 \\ 30 & 40 \\ 50 & 60 \end{bmatrix}$

Add Column by Column

$$\left[\begin{array}{c|c} 1+10 & 2+20 \\ 3+30 & 4+40 \\ 5+50 & 6+60 \end{array}\right] = \left[\begin{array}{c|c} 11 & 22 \\ 33 & 44 \\ 55 & 66 \end{array}\right]$$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 10 \begin{bmatrix} 7 & 8 & 9 \\ 7 & 8 & 9 \end{bmatrix}$

Multiply Each column

$$\left[\begin{array}{c|c|c} 1+10(7) & 2+10(8) & 3+10(9) \\ 4+10(7) & 5+10(8) & 6+10(9) \end{array}\right] = \begin{bmatrix} 71 & 82 & 93 \\ 74 & 85 & 96 \end{bmatrix}$$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Not Enough columns to add

DNE, undefined

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \left(3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ says 3 of first column, 6 of second, but how many of third? DNE, undefined

(e) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \left(3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3+12 \\ 9+24 \\ 15+36 \end{bmatrix} = \begin{bmatrix} 15 \\ 33 \\ 51 \end{bmatrix}$$

(f) $\begin{bmatrix} 1 & a & a \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$ (for a an arbitrary real number)

$A = \begin{bmatrix} 1 & a & a \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ Solve $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2$ $A\vec{x}_3 = \vec{e}_3$ to get $A^{-1} = [\vec{x}_1 | \vec{x}_2 | \vec{x}_3]$. Can use one RREF

$$\left[\begin{array}{ccc|ccc} 1 & a & a & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 - R_3]{R_1 - aR_3} \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -a \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - aR_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\vec{e}_1$ $\vec{e}_2$ $\vec{e}_3$ $\qquad$ $\vec{x}_1$ $\vec{x}_2$ $\vec{x}_3$

$$A^{-1} = \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Convert between a description of a linear transformation and its matrix.

(a) If $T : \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation satisfying $T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T(\vec{\mathbf{e}}_2) = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

then what is $T(\vec{\mathbf{v}})$ if $\vec{\mathbf{v}} = 10\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2$?

$$T(\vec{v}) = T(10\vec{e}_1 + \vec{e}_2) = 10(T(\vec{e}_1)) + T(\vec{e}_2)$$

$$= 10\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 15 \\ 26 \\ 37 \\ 48 \end{bmatrix}$$

fifteen
twenty six
thirty seven
forty eight

(b) Find a matrix A such that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}} \in \mathbb{R}^2$, where T is from part (a).

$$A = \left[A\vec{e}_1 \,\middle|\, A\vec{e}_2 \right] = \left[\begin{array}{c|c} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{array}\right]$$

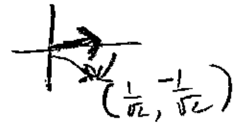
(c) If $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation defined by $S(\vec{\mathbf{x}}) = B\vec{\mathbf{x}}$ for $B = \begin{bmatrix} 1 & 10 & 100 \\ 2 & 20 & 200 \end{bmatrix}$, what is $S(\vec{\mathbf{e}}_2)$?

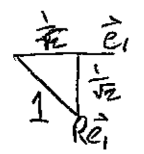
$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad B\vec{e}_2 = 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 10 \\ 20 \end{bmatrix} + 0\begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$B\vec{e}_2 = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$ the second column of B

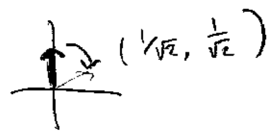
(d) If $R : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors 45 degrees clockwise, then find a matrix C so that $R(\vec{\mathbf{v}}) = C\vec{\mathbf{v}}$ for every $\vec{\mathbf{v}}$ in $\mathbb{R}^2$. **Hint:** What is $R(\vec{\mathbf{e}}_1)$?

$$C = \left[C\vec{e}_1 \,\middle|\, C\vec{e}_2 \right] \qquad C\vec{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$





$$C\vec{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$



$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

3. For $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

Find a matrix A satisfying the given properties:

(a) $A^2 = 0$ but $A \neq 0$ (Challenge: A has no zero entries)

① $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix}$

maybe $d=-a$? maybe $b=a$? $c=d$?

② $A = \begin{bmatrix} \sqrt{73} & \sqrt{73} \\ -\sqrt{73} & -\sqrt{73} \end{bmatrix}$, $A^2 = \begin{bmatrix} 73-73 & 73-73 \\ -73+73 & -73+73 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ a free

(b) $AB = 0$ but $BA \neq 0$ (Challenge: find all A)

$AB = \left[A\vec{b}_1 \mid A\vec{b}_2 \mid A\vec{b}_3 \right] = \left[\vec{a}_1 - \vec{a}_3 \mid -\vec{a}_1 + \vec{a}_2 \mid -\vec{a}_2 + \vec{a}_3 \right] = \left[\vec{0} \mid \vec{0} \mid \vec{0} \right]$

need $\vec{a}_1 = \vec{a}_3$ $\quad \vec{a}_2 = \vec{a}_1$ $\quad \vec{a}_3 = \vec{a}_2$

so $A = \left[\begin{matrix} x \\ y \\ z \end{matrix} \middle| \begin{matrix} x \\ y \\ z \end{matrix} \middle| \begin{matrix} x \\ y \\ z \end{matrix} \right]$ x free, y free, z free $\quad BA = 0$ iff $x=y=z$ $\quad$ so take $x = 13$, $y = -\pi$, $z = \sqrt{73}$

(c) $AC = CA + \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ (Challenge: find all A)

$A = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$ $\quad AC = \begin{bmatrix} x & z \\ y & w \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & x+z \\ y & y+w \end{bmatrix}$

$CA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} x+y & z+w \\ y & w \end{bmatrix}$

so $x = x+y-1$ $\quad x+z = z+w$

$y = y$ $\quad y+w = w+1$

$\to y = 1$ $\quad \to x = w$

w free, z free

$A = \begin{bmatrix} x & z \\ 1 & x \end{bmatrix}$ say $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

(d) $AD = DA$ but $A \neq 0$, $A \neq I_3$ (Challenge: find all A)

$AD = \left[A\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \middle| A\left[\begin{smallmatrix} 0 \\ 2 \\ 0 \end{smallmatrix}\right] \middle| A\left[\begin{smallmatrix} 0 \\ 0 \\ 3 \end{smallmatrix}\right] \right] = \left[\vec{a}_1 \mid 2\vec{a}_2 \mid 3\vec{a}_3 \right]$ $\quad A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$DA = \left[D\vec{a}_1 \mid D\vec{a}_2 \mid D\vec{a}_3 \right] = \begin{bmatrix} a_{11} & 1a_{12} & 1a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$ $\quad$ say $x = 3, y = 4, z = 5$

so a_{11}, a_{22}, a_{33} free but $a_{12} = a_{13} = a_{23} = a_{21} = a_{31} = a_{32} = 0$

(e) $EA = I_2$ (Challenge: find all A)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{e}_1$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}x + \begin{bmatrix} 0 \\ 2 \end{bmatrix}y + \begin{bmatrix} 0 \\ 0 \end{bmatrix}z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x=1$, $y=0$, z free

" $\quad$ " $= \vec{e}_2$, $\quad$ " $\quad$ " $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x=0$, $y=\frac{1}{2}$, z free

$A = \left[\begin{matrix} 1 \\ 0 \\ z_1 \end{matrix} \middle| \begin{matrix} 0 \\ \frac{1}{2} \\ z_2 \end{matrix} \right]$ z_1, z_2 free, say $z_1 = 322$, $z_2 = 7$ $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 322 & 7 \end{bmatrix}$

4. For each matrix A explain why it is invertible or not.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 6 & 7 & 6 \end{bmatrix}$ $1\vec{v}_1 + 0\vec{v}_2 + (-1)\vec{v}_3 = \vec{0}$, Linearly Dependent Columns

so $A\left(\vec{x} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t\right) = A(\vec{x})$

Not

can never be certain what $\vec{x}$ was

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ $\vec{v}_1 + \vec{v}_3 = 2\vec{v}_2$ $A\left(\vec{x} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t\right) = A(\vec{x})$

Not

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 6 & 7 & 0 \end{bmatrix}$ $\vec{v}_3 = 0\vec{v}_1 + 0\vec{v}_2$ $A\left(\vec{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t\right) = A(\vec{x})$

Not

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 9 \end{bmatrix}$

Inv

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 6 & 7 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 - \frac{7}{9}R_3]{R_1 - \frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 6 & 0 & 0 & 1 & -7/9 \\ 0 & 0 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3/9]{R_1 - \frac{1}{3}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & -2/27 \\ 0 & 6 & 0 & 0 & 1 & -7/9 \\ 0 & 0 & 1 & 0 & 0 & 1/9 \end{array}\right] \xrightarrow{R_2/6} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & -2/27 \\ 0 & 1 & 0 & 0 & 1/6 & -7/54 \\ 0 & 0 & 1 & 0 & 0 & 1/9 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{3} & -2/27 \\ 0 & 1/6 & -7/54 \\ 0 & 0 & 1/9 \end{bmatrix}$$

Easier: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ has its pivots in every column, so RREF will work, even if we don't do the calculations ourselves.

(e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Pivots in each column

Inv

Also $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$ just flips x vs z

So its inverse unflips x vs z.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$