Quiz #4 Do the solutions to $y'' = -y$ form a subspace of $\mathbb{R}^{\mathbb{R}}$?

Solution 1: Subspace Check

① $\vec{0}$? $0'' = 0' = 0$ ✓

② $+$? $(f+g)'' = f'' + g'' = -f - g = -(f+g)$ ✓

③ $\cdot$? $(cf)'' = cf'' = c(-f) = -(cf)$ ✓

Solution 2:

From MA214 we know all solutions are of the form $a \cdot \sin(x) + b \cdot \cos(x)$

That is just span$\{\sin(x), \cos(x)\}$ so always a subspace

Solution 3 (for Tuesday)

Let D be the linear transformation that takes a differentiable function to its derivative

The solutions of $y'' = -y$ must ⓐ have 2 derivatives and ⓑ satisfy $(D^2 + I)(y) = \vec{0}$

So we are just looking for homogeneous solutions to $(D^2 + I)\vec{v} = \vec{0}$

Homogeneous solutions always form a subspace

(Finding a basis for the subspace is one of the main ideas in MA214.)

(4, lasso)

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③ $\cdot$? $(cf)'' = cf'' = cf$ ✓

Solution 2:

$$y = a \cdot e^{x} + b \cdot e^{-x}$$

$\text{Span}\{e^{x}, e^{-x}\}$ always a subspace ✓

Solution 3:

Homogeneous solution / Null space of $D^2 - I$

so always a subspace. ✓

Also $D^2 - I = (D - I)(D + I)$

so we want solutions $D - I$ $(y' = y)$ (e^{x})
and $D + I$ $(y' = -y)$ (e^{-x})

↑

Chapter 5 trickery

carbon-hydrogen-oxygen stoichiometry space) which we've been converting to $\mathbb{R}^n$ right as we write it down. We've used a few others implicitly, and I want them written down:

Function spaces: If X is a set (a thing that can answer the question "is x an element of X?"), then $\mathbb{R}^X$ is the vector space of all real-valued functions on X. If X is the set of locations of the NOAA measurement buoys, then $\mathbb{R}^X$ is a great place to store their temperature readings. If $f, g \in \mathbb{R}^X$ then we represent the value of f at some $x \in X$ as either $f(x)$ or f_x. For example sequences take $X = \{1, 2, \ldots\}$ and f_1 means the value of f at 1. If X is the set of atoms, then $\mathbb{R}^X$ is a great place to store stoichiometry problems (or molecules).

Subspaces: We only use a few vector spaces, but actually life is quite complicated and we'll need just a ton of vector spaces. Instead of creating completely new ones, we just use parts of old ones. A **subspace** W is a subset of a vector space V (subset: a thing answering the question "is x an element of W" but only for x that were already in V) satisfying the three properties for $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in W$ and $c \in \mathbb{R}$: $\vec{\mathbf{0}} \in W$, $\vec{\mathbf{u}} + \vec{\mathbf{v}} \in W$, $c \cdot \vec{\mathbf{v}} \in W$.

Notice a subspace does not get a new type of addition or scalar multiplication. All the vector space properties (a) through (f) follow automatically for subspaces. The subspace requirements are only there so that $+$ and $\cdot$ have the right codomain (the $\vec{\mathbf{0}}$ requirement is actually only there to avoid the silly case where W doesn't contain any vectors).

There are two common types of subspaces:

Subspaces defined by a property: W is defined as all the vectors in V that satisfy a certain property. Pros: very easy to check if Joe Random vector is in W. Cons: must check that W is in fact a subspace AND it can be pretty hard to "list" all the vectors in W.

Example: W is all vectors $\vec{\mathbf{v}} = (x, y)$ in $\mathbb{R}^2$ satisfying $y = 2x$. Show this is a subspace. List all the vectors.

① $(0,0)$ has $x=0$, $y=0$, and $2x=0=y$ ✓

$\vec{v} = (a, 2a)$ for any a

② $(a,2a)+(b,2b) = (a+b, 2a+2b)$ has $x=a+b$, $y=2a+2b$, $2x=2(a+b) = 2a+2b = y$ ✓

③ $c(a,2a) = (ac, 2ac)$ has $x=ac$, $y=2ac$, $2x=2ac=y$ ✓

Non-example: W is all vectors $\vec{\mathbf{v}} = (x, y)$ in $\mathbb{R}^2$ satisfying $x^2 + y^2 \leq 1$. Show this is not a subspace.

① $(0,0)$ has $x=0$, $y=0$, $x^2+y^2 = 0 \leq 1$ ✓

② $(a,b)+(c,d)$ has $x=a+c$, $y=b+d$, $x^2+y^2 = \underbrace{a^2+b^2}_{\leq 1}+c^2+d^2+2ac+2ad \leq 2+?1$

try $a=1, b=0, c=1, d=0$ $\quad (1,0)+(0,1)=(1,1)$ but $1^2+1^2 = 2 \not\leq 1$ no!

Subspaces as spans: W is defined as all the linear combinations of some given vectors $\vec{\mathbf{v}}_1, \ldots$. Pros: always a subspace, don't need to check. Cons: hard to check if Joe Random vector is in W, and some technical difficulties listing all vectors only *once*.

Example: W is all linear combinations of $\vec{\mathbf{v}} = (1, -1, 1, 0)$ and $\vec{\mathbf{w}} = (-3, 1, 0, 1)$ in $\mathbb{R}^4$. Write down a simple test to see if (x, y, z, t) is in W. Describe how to list all vectors in W exactly once.

$$\text{Span}(\vec{v}, \vec{w}) = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Test: $x = z - 3t$ and $y = t - z$

List

Same example: W is all linear combinations of the columns of $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & -3 \\ -3 & -4 & 4 \\ -1 & -2 & 1 \end{bmatrix}$. Show that this is actually the same subspace.

Use the test to check that all 3 vectors are in $\text{Span}(\vec{v}, \vec{w})$

$\begin{bmatrix} 0 \\ 2 \\ -3 \\ -1 \end{bmatrix}$ $\quad x=0,\ y=2,\ z=-3,\ t=-1$ $\quad z-3t = -3-3(-1) = 0 = x$ ✓ $\quad t-z = -1-(-3) = 2 = y$ ✓ etc.

Since the span of the columns of A is at least 2 dimensional (not all multiples of the same vector) and $\text{Span}(\vec{v}, \vec{w})$ is 2 dimensional they are equal.