So far, we've always treated our vectors as if they are columns of numbers. This is convenient sometimes but can be artificial and constraining in other circumstances.

Geographic vectors: Consider the following type of vector: at each of the 1600 NOAA approved measurement station, at every hour, the wind speed and atomspheric temperature are measured. One might wish to consider a sort of global temperature vector with one entry for each measurement station. A linear transformation might be an averaging operator from $\mathbb{R}^{1600} \to \mathbb{R}$, but what order should the stations go in? Obviously it doesn't matter in any meaningful way, but it definitely matters in a let's-actually-compute-this way (since the measurement stations are not evenly spread out, we need to use a weighted average, but we need to know which station is where in order to calculate its weight).

Sequence vectors: Another more serious problem is the following type of vector: a repeating signal is formed by superimposing pure signals $\alpha \sin(2\pi\omega t)$ for various amplitudes α and frequencies ω. The frequencies are non-negative integers. So each signal can be represented by a sequence $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ where $\omega_n = n$ and the signal is the infinite sum $\sum \alpha_n \sin(2\pi n t)$ [technically there are also cosines for α_{-n}]. By applying a high bandpass filter we can ignore α_n for larger n, but if we aren't careful this can severely distort the signal. To be able to handle some pretty common signals (even morse code) without pretty severe distortions, we actually need all the α. But how do we write down infinitely many numbers in a column? Especially if we include the cosines too?

Another common version is polynomials. We can store a polynomial by writing down its coefficients, but its not like there is a "highest degree" for polynomials, so the length of our vectors (which should be the same for all the vectors we are using) is unbounded.

Even bigger vectors: What if we allow fractions of polynomials? You might recall "partial fraction" from calculus lets us right down any fraction of polynomials in a unique fashion: for every monic irreducible polynomial f of degree at most 2 and every positive integer n, there is a number $r_{f,n}$ such that our fraction p/q is the infinite sum of $\frac{r_{f,n}}{f^n}$. For each f we get an infinite sequence of $r_{f,n}$. How many f are there? Well, at the very least there is one for every real number: $(x-1)$, $(x-2)$, $(x-3)$, $(x-\pi)$, $(x-\sqrt{10})$, ... Even if you think infinitely long columns are ok, I hope you agree these columns are a wee bit too infinite. Can't we just write down the coefficients of the numerator and denominator? If we did that, then addition would be unfathomably weird (also $\frac{1}{2} = \frac{2}{4}$ gets to be a very serious practical problem; research papers are written on when to "simplify").

Axiom method: In math, we often try to avoid locking ourselves in to a very specific implementation decision. We want our techniques to work in a wide range of situations. Which situations? The axioms answer this by stating a required interface.

A **vector space** is a collection V of "vectors" and operations $+ : V \times V \to V$ and $\cdot : \mathbb{R} \times C \to V$ called addition and scalar multiplication that satisfy the following properties (for any $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$ and $a, b, c \in \mathbb{R}$): (a) $\vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}}$, (b) $0 \cdot \vec{\mathbf{u}} = 0 \cdot \vec{\mathbf{v}}$, (c) $1\vec{\mathbf{v}} = \vec{\mathbf{v}}$, (d) $a \cdot (b \cdot \vec{\mathbf{v}}) = (ab) \cdot \vec{\mathbf{v}}$, (e) $(a+b) \cdot \vec{\mathbf{v}} = (a \cdot \vec{\mathbf{v}}) + (b \cdot \vec{\mathbf{v}})$, (f) $a \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}}) = a \cdot \vec{\mathbf{u}} + a \cdot \vec{\mathbf{v}}$.

From those properties we get that $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$, $\vec{\mathbf{0}} = 0 \cdot \vec{\mathbf{v}}$ satisfies $\vec{\mathbf{0}} + \vec{\mathbf{w}} = \vec{\mathbf{w}}$ and $a \cdot \vec{\mathbf{0}} = \vec{\mathbf{0}}$, and $-\vec{\mathbf{v}} = (-1) \cdot \vec{\mathbf{v}}$ satisfies $\vec{\mathbf{v}} - \vec{\mathbf{v}} = \vec{\mathbf{0}}$.

A vector space is a very serious thing. It needs to clearly define the vectors, how to add them, how to multiply them by scalars, and you have to check those properties for every combination of vectors and scalars. We tend to have only a few vector spaces around. So far in our class we've had the n-dimensional column vectors $V = \mathbb{R}^n$ and some chemistry vector spaces (like the carbon-hydrogen-oxygen stoichiometry space) which we've been converting to $\mathbb{R}^n$ right as we write it down. We've used a few others implicitly, and I want them written down:

Function spaces: If X is a set (a thing that can answer the question "is x an element of X?"), then $\mathbb{R}^X$ is the vector space of all real-valued functions on X. If X is the set of locations of the NOAA measurement buoys, then $\mathbb{R}^X$ is a great place to store their temperature readings. If $f, g \in \mathbb{R}^X$ then we represent the value of f at some $x \in X$ as either $f(x)$ or f_x. For example sequences take $X = \{1, 2, \ldots\}$ and f_1 means the value of f at 1. If X is the set of atoms, then $\mathbb{R}^X$ is a great place to store stoichiometry problems (or molecules).

Subspaces: We only use a few vector spaces, but actually life is quite complicated and we'll need just a ton of vector spaces. Instead of creating completely new ones, we just use parts of old ones. A **subspace** W is a subset of a vector space V (subset: a thing answering the question "is x an element of W" but only for x that were already in V) satisfying the three properties for $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in W$ and $c \in \mathbb{R}$: $\vec{\mathbf{0}} \in W$, $\vec{\mathbf{u}} + \vec{\mathbf{v}} \in W$, $c \cdot \vec{\mathbf{v}} \in W$.

Notice a subspace does not get a new type of addition or scalar multiplication. All the vector space properties (a) through (f) follow automatically for subspaces. The subspace requirements are only there so that $+$ and $\cdot$ have the right codomain (the $\vec{\mathbf{0}}$ requirement is actually only there to avoid the silly case where W doesn't contain any vectors).

There are two common types of subspaces:

Subspaces defined by a property: W is defined as all the vectors in V that satisfy a certain property. Pros: very easy to check if Joe Random vector is in W. Cons: must check that W is in fact a subspace AND it can be pretty hard to "list" all the vectors in W.

Example: W is all vectors $\vec{\mathbf{v}} = (x, y)$ in $\mathbb{R}^2$ satisfying $y = 2x$. Show this is a subspace. List all the vectors.

Non-example: W is all vectors $\vec{\mathbf{v}} = (x, y)$ in $\mathbb{R}^2$ satisfying $x^2 + y^2 \leq 1$. Show this is not a subspace.

Subspaces as spans: W is defined as all the linear combinations of some given vectors $\vec{\mathbf{v}}_1, \ldots$. Pros: always a subspace, don't need to check. Cons: hard to check if Joe Random vector is in W, and some technical difficulties listing all vectors only *once*.

Example: W is all linear combinations of $\vec{\mathbf{v}} = (1, -1, 1, 0)$ and $\vec{\mathbf{w}} = (-3, 1, 0, 1)$ in $\mathbb{R}^4$. Write down a simple test to see if (x, y, z, t) is in W. Describe how to list all vectors in W exactly once.

Same example: W is all linear combinations of the columns of $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & -3 \\ -3 & -4 & 4 \\ -1 & -2 & 1 \end{bmatrix}$. Show that this is actually the same subspace.

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Column spaces

1. $\mathbb{R}^2$ is the set of all points (x, y) in the plane with a specified origin $\vec{\mathbf{0}} = (0, 0)$. The parabola $y = x^2$ is very important in mathematics. Is it a subspace?

2. $\mathbb{R}^3$ is the set of all points (x, y, z) in space with a specified origin $\vec{\mathbf{0}} = (0, 0, 0)$. W is the span of the columns of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{bmatrix}$. Is it a subspace? Can you list the vectors in it?

Function spaces

3. $\mathbb{R}^{\mathbb{R}}$ is all functions whose domain is the real numbers. Show that polynomial functions are a subspace. Make sure to clearly state what $\vec{\mathbf{0}}$ is.

4. The differential equation $y' = y$ specifies a requirement on functions in $\mathbb{R}^{\mathbb{R}}$ (namely, that they be differentiable and that their derivative is equal to themselves). Does this define a subspace? Can you "list" the vectors in it?