4.0 Vocabulary

This is just a quick set of notes on some technical math vocabulary I'd like to be able to use in class. It's hard to deal with function spaces without some fancy language now and then.

A **set** is a mathematical object whose sole property is a "containment predicate." For any other mathematical object x, a set Y must answer the question "is x in Y?" In symbols, this is written $x \in Y$ if the answer is yes, and $x \notin Y$ if the answer is no. If $x \in Y$ we say that "x is in Y" and that "x is an element of Y" and that "Y contains x."

Example: The set of **real numbers** is denoted by $\mathbb{R}$. It answers the question yes if you give it a real number, and no otherwise. For example, $2+2 \in \mathbb{R}$ and $\lim_{x\to\infty} 1/x \in \mathbb{R}$, but $\sqrt{-1} \notin \mathbb{R}$ and $\lim_{x\to\infty} x^2 \notin \mathbb{R}$.

A **function** $f : X \to Y$ is a mathematical object with three properties: its "domain," its "codomain," and its "rule". The domain X and codomain Y are sets. The rule is a mathematical object that must answer the question "what is $f(x)$?" whenever $x \in X$. Furthermore, its answer $f(x)$ must satisfy $f(x) \in Y$ and must only depend on x (if you ask the rule multiple times for the same x, it must always give the same answer $f(x)$).

The **set of all functions** with domain X and codomain Y is written as Y^X. We often abbreviate $\{1, 2, 3, \ldots, n\}$ as just n. So that Y^n means functions from $\{1, 2, 3, \ldots, n\}$ to Y.

Example: For example $\mathbb{R}^3$ is all functions $\vec{\mathbf{v}} : \{1, 2, 3\} \to \mathbb{R}$ where the value $\vec{\mathbf{v}}(1)$ of the function $\vec{\mathbf{v}}$ at the input $x = 1$ is labelled v_1. In other words $\left[\begin{smallmatrix}7\\8\\9\end{smallmatrix}\right]$ is the function f with values $f(1) = 7$, $f(2) = 8$, $f(3) = 9$.

Example: A **vector field** is a function whose domain is some cool thing like the Earth and whose codomain is some vector space like the collection of possible wind velocities. The function describes the velocity of the wind, more specifically, at any location x on the Earth, it describes the velocity $f(x)$ of the wind at that location.

The **direct product** of two sets $X \times Y$ is a set that contains exactly things of the form (x, y) for $x \in X$ and $y \in Y$. You can also view it as the set of functions from $\{1, 2\}$ into the combined set of $X \cup Y$ where $f(1) \in X$ and $f(2) \in Y$.

An **operation** is basically a function, but we are a little more flexible on how it is written. An infix binary operation like vector addition on a vector space V is a function with domain $V \times V$ and codomain V. Instead of writing $+(\vec{\mathbf{v}}, \vec{\mathbf{w}})$ we usually write it in infix notation, $\vec{\mathbf{v}} + \vec{\mathbf{w}}$. The "operation" part mostly just says that if $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are vectors, so is $\vec{\mathbf{v}} + \vec{\mathbf{w}}$ (that is the "codomain" part of the definition of function).

A **subset** of a set Z is a set Y (abbreviated $Y \subseteq Z$) that answers no whenever Z answers no. In other words, if $x \notin Z$ then $x \notin Y$, or more positively, if $x \in Y$ then $x \in Z$. The integers are a subset of the real numbers, and the real numbers are a subset of the complex numbers. Polynomials are a subset of the continuous functions which are a subset of all real-valued functions.

There are not very many ways to specify a set by itself. They are usually big deals. For small ones, you can just list everything between curly braces { and }. Like the set of positive integers whose square is a one digit decimal number is $\{1, 2, 3\}$.

Specifying subsets is easier: $Z = \{x \in Y : f(x)\}$ means the subset of Y where $x \in Z$ means that not only is $x \in Y$ but also the function $f : Y \to \{\texttt{true}, \texttt{false}\}$ returns true for $f(x)$. Usually functions like $f(x)$ are written as equations or inequalities, sometimes with quantifiers. Like $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ means the filled in circle of radius sitting in the plane $\mathbb{R}^2$. $\frac{d}{dx} \notin D$ since $\frac{d}{dx} \notin \mathbb{R}$ (it is a differential operator, not a number), $(2, 3) \notin D$ since $2^2 + 3^2 = 13 \not\leq 1$. However $(1/\pi, 1/10) \in D$ since $1/\pi \in \mathbb{R}$ and $1/10 \in \mathbb{R}$ and $1/\pi^2 + 1/10^2 \leq 1/9 + 1/100 \leq 1$.

4.1 Vector spaces

A **vector space** is a set V (whose elements $\vec{\mathbf{v}}$ are called vectors) with two operations $+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$ that satisfy property (a) through (f) below. For $a, b, c \in \mathbb{R}$ and $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$, we usually write:
$+(\vec{\mathbf{v}}, \vec{\mathbf{w}})$ as $\vec{\mathbf{v}} + \vec{\mathbf{w}}$,
$\cdot(a, \vec{\mathbf{v}})$ as $a\vec{\mathbf{v}}$,
$\cdot(0, \vec{\mathbf{v}})$ as $\vec{\mathbf{0}}$ (there is an axiom that says this vector does not depend on $\vec{\mathbf{v}}$),
$\cdot(-1, \vec{\mathbf{v}})$ as $-\vec{\mathbf{v}}$, and
$+(\vec{\mathbf{v}}, \cdot(-1, \vec{\mathbf{w}}))$ as $\vec{\mathbf{v}} - \vec{\mathbf{w}}$.

(1) $\vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}}$
(2) $0\vec{\mathbf{u}} = 0\vec{\mathbf{v}}$ (the common value is written $\vec{\mathbf{0}}$)
(3) $1\vec{\mathbf{v}} = \vec{\mathbf{v}}$
(4) $a(b\vec{\mathbf{v}}) = (ab)\vec{\mathbf{v}}$
(5) $a(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = a\vec{\mathbf{v}} + a\vec{\mathbf{w}}$
(6) $(a + b)\vec{\mathbf{v}} = a\vec{\mathbf{v}} + b\vec{\mathbf{v}}$

One also has $\vec{\mathbf{v}} - \vec{\mathbf{v}} = \vec{\mathbf{0}}$, $\vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{v}}$, and $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \vec{\mathbf{w}} + \vec{\mathbf{v}}$.

Example Basically the only vector spaces we use are $\mathbb{R}^X$ (or $\mathbb{C}^X$ in small part of chapter 5). However, these are very general.

A **subspace** of a vector space V is a subset W (abbreviated $W \leq V$) that is also a vector space if we use the same $+$ and $\cdot$ from V. It turns out one only needs to check three things

(0) $\vec{\mathbf{0}} \in W$
(+) If $\vec{\mathbf{v}} \in W$ and $\vec{\mathbf{w}} \in W$, then $\vec{\mathbf{v}} + \vec{\mathbf{w}} \in W$
($\cdot$) If $\vec{\mathbf{v}} \in W$ and $a \in \mathbb{R}$, then $a\vec{\mathbf{v}} \in W$

The first one avoids a technical problem ("no vectors" is not a subspace, it breaks a ton of formulas that work otherwise; if you want a basically "nothing" vector space use the single vector $\vec{\mathbf{0}}$). The second one says that if we make the domain of $+$ only be W, then we can also make its codomain only be W. The third one says that if we make the domain of $\cdot$ be W, then we can also make its codomain only be W. Once this is checked, properties (a) through (f) are automatic (they don't mention V or W except to say that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ are in them).

Correction In truth, the only vector spaces we use are subspaces of $\mathbb{R}^X$.

There are two main ways to specify a subspace – by specifying a requirement on the vectors to "winnow", or by giving a few "seed" vectors.

Subspace by property: We can specify a subspace the same way we specify subsets: $W = \{(x, y) \in \mathbb{R}^2 : y = 2x\}$. These are nice because it is pretty easy to check if $\vec{\mathbf{v}} \in W$ (if the property in W is reasonable at all).

Subspace by span: We can also give a set of vectors $S \subseteq V$ (so all contained in one vector space V) and define a subspace $\langle S \rangle$ to be the set whose elements are the linear combinations of (finitely many at a time) vectors in S.

4.2 standarizing a little

A **linear transformation** is a function $T : V \to W$ whose domain and codomain are vector spaces and whose rule satisfies the properties:

(0) $T(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$
(+) $T(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = T(\vec{\mathbf{u}}) + T(\vec{\mathbf{v}})$
($\cdot$) $T(a\vec{\mathbf{v}}) = aT(\vec{\mathbf{v}})$

The (0) property is actually automatic from the ($\cdot$) property and the definition of $\vec{\mathbf{0}}$ (in both V and W) since $T(\vec{\mathbf{0}}) = T(0\vec{\mathbf{v}}) = 0T(\vec{\mathbf{v}}) = \vec{\mathbf{0}}$. However, I thought it might make it easier to remember.

The absolute most standard form of the subspace by property is the **null space**: if $T : V \to W$ is a linear transformation, then $\mathrm{Nul}(T) = \{\vec{\mathbf{v}} \in V : T(\vec{\mathbf{v}}) = \vec{\mathbf{0}}\}$.

The absolute most standard form of the subspace by span is the **column space**: if $T : V \to W$ is a linear transformation, then $\mathrm{Col}(R)$ is the span of $S = \{T(\vec{\mathbf{v}}) : \vec{\mathbf{v}} \in V\}$. If S' spans V, then you can just use $S = \{T(\vec{\mathbf{v}}) : \vec{\mathbf{v}} \in S'\}$. When $V = \mathbb{R}^n$, you can use $S = \{T(\vec{\mathbf{e}}_i) : i = 1, 2, \ldots, n\}$ which are just the columns of the matrix of T, hence the name.

Every subspace of V is the null space of some linear transformation $T : V \to W$, and every subspace is the column space of some linear transformation $T : U \to V$.

The null space formulation expresses the subspace as the vectors satisfying some equations ($T(\vec{\mathbf{v}}) = \vec{\mathbf{0}}$ can usually be written as some very simple equations). The column space formulation expresses the subspace as what you get if you start with a few vectors.

4.4 seriously standardizing

Every subspace $W \leq V$ is the column space of $T : \mathbb{R}^X \to V$ where T is both 1-1 and onto. In other words, for every vector $\vec{\mathbf{w}} \in W$ there is precisely one vector $\hat{\vec{\mathbf{w}}} = \sum w_x \vec{\mathbf{e}}_x \in \mathbb{R}^X$ with $T(\hat{\vec{\mathbf{w}}}) = \vec{\mathbf{w}}$ (and $T(\vec{\mathbf{x}})$ is always a vector in W). The numbers w_x (where we keep track of the x; this literally just means the function $\hat{\vec{\mathbf{w}}}$) are called the coordinates of $\vec{\mathbf{w}}$ with respect to T (or sometimes, "with respect to the basis $\{T(\vec{\mathbf{e}}_x)\}$").

This lets us take whatever weird vectors are in W and treat them as collections of numbers. If $X = \{1, 2, \ldots, n\}$, then we can imagine that vectors in W are just column vectors (they aren't; $\vec{\mathbf{w}}$ is some weird vector, but it is also $T(\hat{\vec{\mathbf{w}}})$ where $\hat{\vec{\mathbf{w}}}$ is a column vector).

For example $3\sin(x) + \sqrt{2}\cos(x)$ is not a column vector, but in the vector subspace $W = \{f : f'' + f = 0\}$ of $\mathbb{R}^{\mathbb{R}}$ consisting of solutions to the differential equation $y'' = -y$ it could be represented by $\left[\begin{smallmatrix} 3 \\ \sqrt{2} \end{smallmatrix}\right]$ with respect to the basis $\{\sin(x), \cos(x)\}$ or the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^{\mathbb{R}}$ defined by $T(a, b) = a\sin(x) + b\cos(x)$.

MA322-007 Mar 3 Quiz Name:__________________________

HW#4. Is the following a subspace? Show the subspace check.
$W = \{(x, y) : 3x + 2y = 5\} \subseteq \mathbb{R}^2$

HW#4' Is the following a subspace? Show the subspace check.
$W = \{(x, y) : 3x + 2y = 0\} \subseteq \mathbb{R}^2$

HW#5 Is the following a subspace? Show the subspace check.
$W = \{t \mapsto at^2 : a \in \mathbb{R}\} \subseteq \mathbb{R}^{\mathbb{R}}$

HW#6 Is the following a subspace? Show the subspace check.
$W = \{t \mapsto t^2 + a : a \in \mathbb{R}\} \subseteq \mathbb{R}^{\mathbb{R}}$

HW#13 Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(a) How many vectors are in $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$? Is $\vec{\mathbf{w}} \in \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$?

(b) How many vectors are in $\mathrm{Span}(\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\})$? Is $\vec{\mathbf{w}} \in \mathrm{Span}(\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\})$?

4.2 Write the null space of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as the span of the columns of a matrix N.