For 3b on the practice exam, it'd be nice to directly use RREF to find the answer. My solution on 3b wrote it as a column space, and then used RREF.

This new solution does the same thing, but is a little sneaky about it, so maybe it'll be helpful to see the variety of things RREF can do.

We want to figure out which vectors can be written in the form $\begin{bmatrix} a+b \\ c-d \\ a+c \\ b-d \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$ are **any** real numbers. . . that sum to 0. It'd be nice to write the vector as $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ and just have some equations that x, y, z, w need to satisfy.

So let's do that!

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \begin{array}{rcl} x &=& a+b \\ y &=& c-d \\ z &=& a+c \\ w &=& b-d \end{array}, a+b+c+d=0 \right\}$$

So easy! Except wait. What are a, b, c, d now? Free variables. Oh, so x, y, z, w are not free variables and though they are related to each other, the only relations we wrote down are between them and these free variables a, b, c, d. What is x? "It is a number that can be written as $a - b$ for some a, b." Can't any number be written that way? Just take $b = 0$. "Well, sure, but then you've chosen b, and that affects y, z, w as well."

Oh man, what a terrible answer!

Never fear. RREF can fix this..

$$\left[\begin{array}{cccccccc|c} a & b & c & d & x & y & z & w & \\ \hline 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccccccc|c} a & b & c & d & x & y & z & w & \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & -3/2 & -1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right]$$

The first four rows tell us that $a = -y + 1.5z + 0.5w$, $b = -0.5z + 0.5w$, $c = \ldots$, $d = \ldots$ which would be nice, except we don't really care about a, b, c, d. The last row is the only one that concerns us: $x = z + w - y$ (or $x + y = z + w$). That and the fact that y, z, w are free variables!

$$\text{So } W = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : x + y = z + w \right\} = \operatorname{Nul}\left(\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \right).$$

Done!

(And you can check that indeed $(a+b)+(c-d)$ does equal $(a+c)+(b-d)$ no matter what values of a, b, c, d you chose. Since $d = -a-b-c$ can be eliminated, we have a 3-dimensional space W living in a 4-dimensional world $\mathbb{R}^4$, so we need one equation to define it. In other words, a 1-row matrix B for the null space.)