MA322-007 Mar 12 Exam Name: Key

1. Are these subspaces? Show the subspace check.

(a) $W = \{(x, y) \in \mathbb{R}^2 : y = (x-1)^2 - x^2\}$ No

☹ $\vec{0}$: $\vec{0} = (0,0)$ so $x=0$, $y=0$ but $\overset{y}{0} \neq (\overset{x}{0}-1)^2 - \overset{x}{0}^2 = 1$

(b) $W = \left\{ \begin{bmatrix} a \\ b \\ 5a+7b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ Yes $W = \mathrm{Col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 5 & 7 \end{bmatrix} \right)$

$= \mathrm{Nul}\left(\begin{bmatrix} 5 & 7 & -1 \end{bmatrix} \right)$

☺ (0) $a=b=0$ works ✓

☺ (+) $\begin{bmatrix} x \\ y \\ 5x+7y \end{bmatrix} + \begin{bmatrix} r \\ s \\ 5r+7s \end{bmatrix} = \begin{bmatrix} x+r \\ y+s \\ 5x+7y+5r+7s \end{bmatrix} = \begin{bmatrix} (x+r) \\ (y+s) \\ 5(x+r)+7(y+s) \end{bmatrix}$ ✓

☺ (·) $c \begin{bmatrix} x \\ y \\ 5x+7y \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ c(5x+7y) \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ 5(cx)+7(cy) \end{bmatrix}$ ✓

(c) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 5x + 7y - z = 0 \right\}$ Yes $W = \mathrm{Nul}\left(\begin{bmatrix} 5 & 7 & -1 \end{bmatrix} \right)$

☺ (0) $x = 0 = y = z$ works $\left(5(0) + 7(0) - (0) = 0 \checkmark \right)$

☺ (+) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$ if $5a + 7b - c = 0$ and $5d + 7e - f = 0$

then $5(a+d) + 7(b+e) - (c+f) = 0 + 0 = 0$ ✓

☺ (·) $c \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ if $5(x) + 7(y) - z = 0$ then $5(cx) + 7(cy) - (cz)$

$= c(5x + 7y - z) = c \cdot 0 = 0$ ✓

(d) $W = \left\{ \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix} \in \mathbb{R}^3 : x + y = 1 \right\}$ No

☹ $\vec{0}$ $\begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ means $x=0, y=0$, but $\overset{x}{0} + \overset{y}{0} = 0 \neq 1$

2. $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_4 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(a) Which of these vectors are in Nul(A)? $\boxed{\vec{v}_4 \text{ only}}$

$A\vec{v}_1$ DNE, so no

$A\vec{v}_2$ DNE, so no

$$A\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 12 \\ 16 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{so no}$$

$$A\vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \quad \text{so yes}$$

(b) Which of these vectors are in Col(A)? Only $\vec{v}_2$ $\qquad V_2 = A\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Solve $A\vec{x} = \vec{v}_i$ using RREF

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 3 & 5 & 0 & 1 \\ 2 & 4 & 6 & 1 & 0 \\ & & & v_1 & v_2 \end{array}\right] \xrightarrow[R_4 - 2R_1]{R_3 - R_1} \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & \underset{\uparrow \text{ so not } \vec{v}_1}{\boxed{-1}} & 0 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|cc} x & y & z & v_1 & v_2 \\ 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{-1} & \boxed{0} \\ 0 & 0 & 0 & \underset{\uparrow \text{ So not } \vec{v}_1}{\boxed{-1}} & \underset{\uparrow \text{ So } \vec{v}_2}{\boxed{0}} \end{array}\right]$$

$$-2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

(c) Give lots of examples (5 to infinitely many) of vectors in Nul(A):

Any multiple of $\vec{v}_4$, $\begin{bmatrix} x \\ -2x \\ x \end{bmatrix}$ for any $x \in \mathbb{R}$

(Those are the only ones.)

(d) Give lots of examples (5 to infinitely many) of vectors in Col(A):

$$\text{Col}(A) = \text{Col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x \\ y \\ x+y \\ 2x \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

3. Convert to nullspaces. Find a matrix B so that $\mathrm{Nul}(B)$ is as required:

(a) $\mathrm{Nul}(B) = \mathrm{Col}(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

$$\mathrm{Col}(A) = \mathrm{Col}\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}\right) = \mathrm{Col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix}\right)$$

Other way RREF

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 2 & b \\ 1 & 3 & 5 & c \\ 2 & 4 & 6 & d \end{array}\right] \xrightarrow[R_4-2R_1]{R_3-R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 2 & b \\ 0 & 1 & 2 & c-a \\ 0 & 0 & 0 & d-2a \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 0 & c-a-b \\ 0 & 0 & 0 & d-2a \end{bmatrix} \text{(Done)} \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & a-2b \\ 0 & 1 & 2 & b \\ 0 & 0 & 0 & c-a-b \\ 0 & 0 & 0 & d-2a \end{array}\right]$$

$$\text{so } A\begin{bmatrix} a-2b \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ as long as } \begin{array}{l} c = a+b \\ d = 2a \end{array}$$

(b) $\mathrm{Nul}(B) = \left\{ \begin{bmatrix} x \\ y \\ 3x+7y \\ 8x-9y \end{bmatrix} : x, y \in \mathbb{R} \right\}$ $= \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \begin{array}{l} z = 3x+7y \\ w = 8x-9y \end{array} \right\}$

$$= \mathrm{Nul}\left(\begin{array}{cccc} x & y & z & w \\ \hline 3 & 7 & -1 & 0 \\ 8 & -9 & 0 & -1 \end{array} \right)$$

4. Convert to column spaces. Find a matrix B so that $\mathrm{Col}(B)$ is as required:

(a) $\mathrm{Col}(B) = \mathrm{Nul}(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

Find Nul(A) using RREF

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -4 & -8 & -12 & 0 \end{array}\right] \xrightarrow{R_2/-4}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \end{array}$$

$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \begin{array}{l} x = z + 2w \\ y = -2z - 3w \\ z \text{ is free} \\ w \text{ is free} \end{array} \right\} = \mathrm{Col}\left(\begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

(b) $\mathrm{Col}(B) = \left\{ \begin{bmatrix} a + 2b \\ 3c - 4d \\ 5a + 6c \\ 7b - 8d \end{bmatrix} : a + b + c + d = 0 \right\}$

$-d = a + b + c$, substitute in

$$= \left\{ \begin{bmatrix} 1a + 2b \\ 4a + 4b + 7c \\ 5a + 6c \\ 8a + 15b + 8c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 1 & 2 & 0 \\ 4 & 4 & 7 \\ 5 & 0 & 6 \\ 8 & 15 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 0 \\ 4 & 4 & 7 \\ 5 & 0 & 6 \\ 8 & 15 & 8 \end{bmatrix} \right)$$