Cat ate my key. Seriously.

MA322-007 Mar 12 Practice Exam

Name: Key

1. Are these subspaces? Show the subspace check.

(a) $W = \{(x, y) \in \mathbb{R}^2 : y = (x-1)^2 - 1\}$ No!

(0) $\vec{0} = (0,0)$ $x=0, y=0$ and $0 = (0-1)^2 - 1$

(+) $(a, (a-1)^2-1) + (b, (b-1)^2-1) = (a+b, (a-1)^2 + (b-1)^2 - 2)$ but $(a+b-1)^2 - 1 \neq (a-1)^2 + (b-1)^2 - 2$

for example $\underset{\in W}{(1,-1)} + \underset{\in W}{(1,-1)} = (2,-2)$ but $-2 \neq (2-1)^2 - 1 = 0$

(b) $W = \left\{ \begin{bmatrix} a-2b \\ b-2a \\ a+b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ Yes

Easier $W = \mathrm{Col}\left(\begin{bmatrix} 1 & -2 \\ -2 & 1 \\ 1 & 1 \end{bmatrix}\right)$

(0) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ take $a=0, b=0$ for example

(+) $\begin{bmatrix} a-2b \\ b-2a \\ a+b \end{bmatrix} + \begin{bmatrix} c-2d \\ d-2c \\ c+d \end{bmatrix} = \begin{bmatrix} (a+c)-2(b+d) \\ (b+d)-2(a+c) \\ (a+c)+(b+d) \end{bmatrix}$ so "a" $= a+c$, "b" $= b+d$

(·) $c\begin{bmatrix} a-2b \\ b-2a \\ a+b \end{bmatrix}$ has "a" $= ca$, "b" $= cb$

(c) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^2 : x + y + z = 2x + 3y = 0 \right\}$ Yes

Easier $W = \mathrm{Nul}\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}\right)$

(0) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has $x=0, y=0, z=0$. $0+0+0 = 2(0)+3(0) = 0$ ✓

(+) if $x_1 + y_1 + z_1 = 2x_1 + 3y_1 = 0$
$+$ and $x_2 + y_2 + z_2 = 2x_2 + 3y_2 = 0$
then $(x_1+x_2) + (y_1+y_2) + (z_1+z_2) = 2(x_1+x_2) + 3(y_1+y_2) = 0+0=0$ ✓

(·) $0 = c0 = c(x_1 + y_1 + z_1) = (cx_1) + (cy_1) + (cz_1)$ etc ✓

(d) $W = \left\{ \begin{bmatrix} a-2b+c \\ b-2a-3c \\ a+b+c \end{bmatrix} : 2a + 3b + 4c = 0 \right\}$

"Easier" is $T: \mathbb{R}^3 \to \mathbb{R}^3 : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} a-2b+c \\ b-2a+3c \\ a+b+c \end{bmatrix}$

$W = T(\mathrm{Nul}([2\ 3\ 4]))$

is the column space restricted to a null space, so a subspace.

($\vec{0}$) $a=0, b=0, c=0$ works

(+) $\begin{bmatrix} a_1-2b_1+c_1 \\ b_1-2a_1-3c_1 \\ a_1+b_1+c_1 \end{bmatrix} + \begin{bmatrix} a_2-2b_2+c_2 \\ b_2-2a_2-3c_2 \\ a_2+b_2+c_2 \end{bmatrix}$ has "a" $= a_1+a_2$, "b" $= b_1+b_2$, "c" $= c_1+c_2$

and if $2a_1 + 3b_1 + 4c_1 = 0$
$+$ and $2a_2 + 3b_2 + 4c_2 = 0$
then $2a + 3b + 4c = 0$ too ✓

(·) Same idea

2. $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $A = \begin{array}{c} \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \end{array}$.

Easy (a) Which of these vectors are in Nul(A)? (Challenge: give the general answer)

$Nul(A) = \{ \vec{x} : A\vec{x} = 0 \} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} x + \begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix} y + \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Not $\vec{v}_1$, not $\vec{v}_2$. (wrong type of vector completely)

$\vec{V}_3$ works since $\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Not $\vec{v}_4$ since $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 \neq \vec{0}$

Hard (b) Which of these vectors are in Col(A)? (Challenge: give the general answer)

$Col(A) = \{ A\vec{x} : \vec{x} \in \mathbb{R}^3 \} = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \\ 10x + 11y + 12z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$

not $\vec{v}_3, \vec{v}_4$ (wrong size)

$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 1 \\ 4 & 5 & 6 & 2 & 2 \\ 7 & 8 & 9 & 3 & 2 \\ 10 & 11 & 12 & 4 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|cc} 1 & 0 & -1 & -1/3 & 0 \\ 0 & 1 & 2 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

so z is free
but $x = -1/3, y = 2/3, z = 0$ gives $\vec{V}_1$
and nothing gives $\vec{V}_2$ since $0x + 0y + 0z \neq 1$

Hard (c) Give lots of examples (5 to infinitely many) of vectors in Nul(A):

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 10 & 11 & 12 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $x = z$, $y = -2z$, z free

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} z$

$Nul(A) = Col\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} \pi \\ -2\pi \\ \pi \end{bmatrix}, \begin{bmatrix} \sqrt{3} \\ -\sqrt{12} \\ \sqrt{3} \end{bmatrix}, etc \right\}$

Easy (d) Give lots of examples (5 to infinitely many) of vectors in Col(A):

Since $\vec{a}_2 = \vec{a}_1 + \vec{a}_3$ we only need $\vec{a}_1, \vec{a}_3$. In fact $\vec{a}_2 - \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\frac{2\vec{a}_1 - a_2}{3} = \begin{bmatrix} 0 \\ 3 \\ 6 \\ 9 \end{bmatrix} \frac{1}{3} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

$Col(A) = \left\{ \begin{bmatrix} x \\ x + y \\ x + 2y \\ x + 3y \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, etc, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, etc \right\}$

3. Convert to nullspaces. Find a matrix B so that $\mathrm{Nul}(B)$ is as required:

(a) $\mathrm{Nul}(B) = \mathrm{Col}(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ -4 & -5 & -6 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 2 & 3 & b \\ 4 & 5 & 6 & c \\ -4 & -5 & -6 & d \end{array}\right] \xrightarrow[R_4+R_3]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 0 & 0 & b-a \\ 4 & 5 & 6 & c \\ 0 & 0 & 0 & c+d \end{array}\right] \xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 0 & 0 & b-a \\ 0 & -3 & -6 & c-4a \\ 0 & 0 & 0 & c-a \end{array}\right] \xrightarrow[R_1-2R_2]{R_3/-3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{5}{3}a-\frac{2}{3}c \\ 0 & 0 & 0 & b-a \\ 0 & 1 & 2 & (c-4a)/3 \\ 0 & 0 & 0 & c+d \end{array}\right]$$

$\leftarrow$ Done

has a solution iff $b-a=0$ and $c+d=0$ $\quad (a=b,\ c=-d)$

(one solution is $x = z-\frac{5}{3}-\frac{2}{3}c$
$y = -2z+(c-4a)/3$
z is free,
but we don't care how we write the vector as a linear combination, only if we can do it at all.)

$$\text{So } \mathrm{Col}(A) = \mathrm{Nul}\left(\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right)$$

(b) $\mathrm{Nul}(B) = \left\{ \begin{bmatrix} a+b \\ c-d \\ a+c \\ b-d \end{bmatrix} : a+b+c+d=0 \right\}$

$d = -a-b-c$

$$\text{so } \mathrm{Nul}(B) = \left\{ \begin{bmatrix} a+b \\ c+a+b+c \\ a+c \\ b+a+b+c \end{bmatrix} : a,b,c \in \mathbb{R} \right\} = \mathrm{Col}\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right)$$

so we can now repeat 3a on it

(c) $\mathrm{Nul}(B) = \left\{ \begin{bmatrix} x \\ y \\ 3x+7y \\ 8x-9y \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \begin{array}{l} z = 3x+7y \\ w = 8x-9y \end{array} \right\}$

$$= \mathrm{Nul}\left(\begin{bmatrix} -3 & -7 & 1 & 0 \\ -8 & +9 & 0 & 1 \end{bmatrix}\right)$$

4. Convert to column spaces. Find a matrix B so that $\mathrm{Col}(B)$ is as required:

(a) $\mathrm{Col}(B) = \mathrm{Nul}(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 6 & 7 & 8 & 9 & 10 & 0 \end{array}\right] \xrightarrow[R_2-6R_1]{RREF} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & -5 & -10 & -15 & -20 & 0 \end{array}\right] \xrightarrow[\substack{R_2/-5 \\ R_1-2R_2}]{} \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \end{array}\right]$$

so
$x_1 = x_3 + 2x_4 + 3x_5$
$x_2 = -2x_3 - 3x_4 - 4x_5$
x_3 is free
x_4 is free
x_5 is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5$$

$$\mathrm{Nul}(A) = \mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

(b) $\mathrm{Col}(B) = \left\{ \begin{bmatrix} a+b \\ c-d \\ a+c \\ b-d \end{bmatrix} : a+b+c+d = 0 \right\}$.

as in 3b, $= \mathrm{Col}\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \right)$

(c) $\mathrm{Col}(B) = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : c = 3a + 5b, d = 7a + 9b \right\} = \left\{ \begin{bmatrix} a \\ b \\ 3a+5b \\ 7a+9b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ 5 \\ 9 \end{bmatrix} b : a, b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 5 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} : \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \right\}$$

$$= \mathrm{Col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 5 \\ 7 & 9 \end{bmatrix} \right)$$